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MULTIVARIATE HYPERGEOMETRIC TERMS

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by

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Abstract

In [15], Wilf and Zeilberger developed a proof theory for hypergeometric multisums centered around the notion of a multivariate hypergeometric term. A multivariate function $f(z_1, \dots, z_k) = f(\mathbf{z})$ from \mathbb{Z}^k to a field K is a *hypergeometric term* if for each $i \in \{1, \dots, k\}$ there exist nonzero polynomials $A_i(\mathbf{z})$ and $B_i(\mathbf{z})$ in $K[\mathbf{z}]$ such that

$$A_i(\mathbf{z})f(\mathbf{z}) = B_i(\mathbf{z})f(\mathbf{z} + \mathbf{e}_i)$$

for all $\mathbf{z} \in \mathbb{Z}^k$. Here $\mathbf{e}_1, \dots, \mathbf{e}_k$ is the standard basis for \mathbb{Z}^k . If f is not a *zero divisor*, then the *term ratios* $R_1 = A_1/B_1, \dots, R_k = A_k/B_k$ are unique and satisfy the relation

$$R_i(\mathbf{z})R_j(\mathbf{z} + \mathbf{e}_i) = R_j(\mathbf{z})R_i(\mathbf{z} + \mathbf{e}_j) \quad \text{for each } i, j \in \{1, \dots, k\}$$

(cf. [14]).

We introduce the concepts of divisibility lattice paths, rational Galois spaces, and fixed factors of rational functions to the study of the relation for the term ratios. We prove that a solution R_1, \dots, R_k must be of the form [Ore–Sato]

$$R_i(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{e}_i)}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{e}_i)} \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{e}_i \cdot \mathbf{v}} \frac{a_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{z} + j)}{b_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{z} + j)} \quad \text{for } i \in \{1, \dots, k\},$$

where C and D are polynomials, V is a finite subset of \mathbb{Z}^k , and, for each $\mathbf{v} \in V$, $a_{\mathbf{v}}$ and $b_{\mathbf{v}}$ are *univariate* polynomials all independent of i . The symbol \prod_j^b denotes $\prod_{i=a}^{b-1}$ if $b > a$, 1 if $a = b$, and $1/\prod_{i=b}^{a-1}$ if $a > b$.

We use this factorization of R_i to answer an obvious question about multivariate hypergeometric terms. Recall the Pochhammer symbol $(m)_r = m(m+1) \cdots (m+r-1)$. The multivariate hypergeometric terms that arise in practice have the form

$$f(\mathbf{z}) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} \frac{C(\mathbf{z})}{D(\mathbf{z})} \frac{\prod_{i=1}^p (m_i)_{\mathbf{v}_i \cdot \mathbf{z} + r_i}}{\prod_{j=1}^q (n_j)_{\mathbf{w}_j \cdot \mathbf{z} + s_j}}$$

where $\gamma_1, \dots, \gamma_k \in K$, C and D are polynomials in $K[\mathbf{z}]$, the \mathbf{v}_i and \mathbf{w}_j are in \mathbb{Z}^k , the r_i and s_j are in \mathbb{Z} , and the m_i and n_j are in K . The question is: do all hypergeometric terms have this form? We prove that if K is algebraically closed, and the hypergeometric term is honest, then such an expression for f exists *piecewise*. This is trivial in the case of one variable, but *not* in the case of several variables. We use this result to settle the discrete part of a conjecture of Wilf and Zeilberger [15] by showing that a holonomic hypergeometric term is piecewise proper, which means roughly that we can take $D(\mathbf{z}) = 1$ in the expression for f above.

For any subsets A and B of an additive group G , define $A + B = \{a + b : a \in A \text{ and } b \in B\}$ and $-A = \{-a : a \in A\}$. A subset S of G is said to be sum-free, complete, and symmetric respectively if $S + S \subset S^c$, $S + S \supset S^c$, and $S = -S$. Cameron asked if for all sufficiently large moduli m there exists a sum-free complete set in $\mathbb{Z}/m\mathbb{Z}$ that is not symmetric [7]. We answer Cameron's question by showing there exists such a set for all moduli greater than or equal to 890626. We also show that every sum-free complete set in $\mathbb{Z}/m\mathbb{Z}$ that is not symmetric can be used to construct a counterexample to a conjecture of Conway disproved by Marica [9]. Conway conjectured that for any finite set S of integers, $|S + S| \leq |S - S|$.

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Chapter 1

Introduction

1.1 Hypergeometric terms and term ratios

Throughout this work K denotes a field of characteristic zero. We write $K[z]$ for the ring of polynomials in one variable over K , and $K[\mathbf{z}] = K[z_1, \dots, z_k]$ for the ring of polynomials in k variables over K . Similarly, we write $K(z)$ for field of rational functions in one variable over K , and $K(\mathbf{z}) = K(z_1, \dots, z_k)$ for the field of rational functions in k variables over K .

Definition 1.1.1 For any function $f: \mathbb{Z}^k \rightarrow K$ and any vector $\mathbf{v} \in \mathbb{Z}^k$, the function $f^{\mathbf{v}}: \mathbb{Z}^k \rightarrow K$ is defined by

$$f^{\mathbf{v}}(\mathbf{z}) = f(\mathbf{z} + \mathbf{v}) = f(z_1 + v_1, \dots, z_k + v_k).$$

Similarly, for any rational function $R \in K(\mathbf{z})$ and any vector $\mathbf{v} \in \mathbb{Q}^k$, the rational function $R^{\mathbf{v}} \in K(\mathbf{z})$ is defined by

$$R^{\mathbf{v}}(\mathbf{z}) = R(\mathbf{z} + \mathbf{v}) = R(z_1 + v_1, \dots, z_k + v_k).$$

Definition 1.1.2 A *hypergeometric term* on \mathbb{Z}^k over a field K is a function $f: \mathbb{Z}^k \rightarrow K$ such that for $i \in \{1, \dots, k\}$ there exist nonzero polynomials A_1, \dots, A_k and $B_1, \dots, B_k \in K[\mathbf{z}]$ such that

$$A_i f = B_i f^{\mathbf{e}_i}.$$

Of course, this last equation is equivalent to $A_i(\mathbf{z})f(\mathbf{z}) = B_i(\mathbf{z})f(\mathbf{z} + \mathbf{e}_i)$ for all $\mathbf{z} \in \mathbb{Z}^k$.

For any hypergeometric term f on \mathbb{Z}^k and any $\mathbf{v} \in \mathbb{Z}^k$, a rational function $R_{\mathbf{v}} \in K(\mathbf{z})$ is a *term ratio in the direction \mathbf{v}* if

$$R_{\mathbf{v}} = \frac{A_{\mathbf{v}}}{B_{\mathbf{v}}}$$

for some nonzero polynomials $A_{\mathbf{v}}, B_{\mathbf{v}} \in K[\mathbf{v}]$ such that $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$.

Definition 1.1.3 A function f on \mathbb{Z}^k is a *zero divisor* if there exists a nonzero polynomial $p \in K[\mathbf{z}]$ such that $pf = 0$.

We show in Lemma 3.1.5 that if a hypergeometric term f is not a zero divisor, then for each $\mathbf{v} \in \mathbb{Z}^k$ there exists a unique term ratio $R_{\mathbf{v}}$ in the direction \mathbf{v} . Furthermore, we show in Lemma 3.1.6 that if f is not a zero divisor then the term ratios satisfy

$$R_{\mathbf{v}}R_{\mathbf{w}}^{\mathbf{v}} = R_{\mathbf{w}}R_{\mathbf{v}}^{\mathbf{w}}$$

for all $\mathbf{w}, \mathbf{v} \in \mathbb{Z}^k$. In particular, letting $R_i = R_{\mathbf{e}_i}$, we have $R_i R_j^{\mathbf{e}_i} = R_j R_i^{\mathbf{e}_j}$ for all $i, j \in \{1, \dots, k\}$ (cf. [14]). This is the relation for the term ratios of a hypergeometric term. The first step in understanding the structure of a hypergeometric term is to understand the structure of rational solutions of this system of equations.

1.2 The rational solutions of $R_i R_j^{\mathbf{e}_i} = R_j R_i^{\mathbf{e}_j}$

It will be useful to define

$$\prod_a^b A_i = \begin{cases} \prod_{i=a}^{b-1} A_i & \text{if } b > a \\ 1 & \text{if } b = a \\ 1 / \prod_{i=b}^{a-1} A_i & \text{if } a > b. \end{cases}$$

The usefulness of the notation stems in part from the fact that

$$\prod_a^b A_i \prod_b^c A_i = \prod_a^c A_i$$

and

$$\prod_a^b A_i = \prod_b^a \frac{1}{A_i},$$

which is easily verified.

In Chapter 2, we prove Theorem 2.8.4 [Ore–Sato] which completely describes the rational solutions of the system of equations

$$R_i R_j^{e_i} = R_j R_i^{e_j} \quad i, j \in \{1, \dots, k\}.$$

Theorem 2.8.4 Let $R_i \in K(z)$, $i = 1, \dots, k$, be rational functions such that

$$R_i R_j^{e_i} = R_j R_i^{e_j} \quad \text{for all } i, j \in \{1, \dots, k\}.$$

Then there exist polynomials C and $D \in K[z]$, a finite set $V \subset \mathbb{Z}^k$, and univariate polynomials a_v and $b_v \in K[z]$ for each $v \in V$ such that for all $i \in \{1, \dots, k\}$,

$$R_i(z) = \frac{C(z + e_i)}{C(z)} \frac{D(z)}{D(z + e_i)} \prod_{v \in V} \prod_j^{v_i} \frac{a_v(z \cdot v + j)}{b_v(z \cdot v + j)}.$$

The proof of Theorem 2.8.4 is quite involved. The first part of the proof is showing that the solution can be expressed in the form

$$R_i = \frac{C^{e_i}}{C} \frac{D}{D^{e_i}} \frac{A_i}{B_i},$$

$i = 1, \dots, k$, where C and D are polynomials (not depending on i) and A_i and B_i are products of *simple* polynomials. A simple polynomial is a composition of a univariate polynomial and a linear polynomial $v \cdot z$. The one-dimensional case is trivial. By using Gosper's Lemma [8], we reduce the two-dimensional case to the following lemma. Let A_1, A_2, B_1 , and $B_2 \in K[z_1, z_2]$. If A_1 and $B_1^{ne_1}$ are relatively prime for all $n \in \mathbb{Z}$, A_2 and B_2 are relatively prime, and $R_1 = A_1/B_1$ and $R_2 = A_2/B_2$ satisfy $R_1 R_2^{e_1} = R_2 R_1^{e_2}$, then the irreducible divisors of A_1, B_1, A_2 , and B_2 are simple. An interesting feature of the proof of the lemma is the association of a lattice path with an irreducible divisor of A_1, B_1, A_2 , or B_2 . (A *lattice path in \mathbb{Z}^k* is a sequence $\{T_i\}_{i \geq 1}$ in \mathbb{Z}^k such that $T_i - T_{i-1} \in \{\pm e_1, \dots, \pm e_k\}$ for all $i > 1$.) The primeness conditions of the lemma impose restrictions of the shape of the lattice path, from which we deduce that the path is unbounded.

From the unboundedness of the path, it follows that d satisfies a nontrivial relation $d = d^v$, and from this it follows that d is simple. We derive the higher-dimensional cases from the two-dimensional case with the aid of the notions of the *rational Galois space* of a rational function and the *fixed factor* of a rational function for a subspace.

The second part of the proof of Theorem 2.8.4 is deriving the formula for R_i given the simplicity of the divisors of the A_i and B_i . A crucial step is observing that the multiplicative components of the R_i determined by rational Galois space still satisfy the relation for the term ratios: if s is a subspace of \mathbb{Q}^k and $R_{i,s} = \text{fix}_s R_i$, then

$$R_{i,s} R_{j,s}^{e_i} = R_{j,s} R_{i,s}^{e_j}.$$

1.3 The multiplicative structure of a hypergeometric term

Define a *hyperplane* in \mathbb{Z}^k to be a set of the form $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} = n\}$, where $n \in \mathbb{Z}$ and \mathbf{v} is some vector in \mathbb{Z}^k . Define a *half-space* in \mathbb{Z}^k to be a set of the form $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} > n\}$, where $n \in \mathbb{Z}$ and \mathbf{v} is some vector in \mathbb{Z}^k . A *set of measure zero* in \mathbb{Z}^k is a set that can be covered by a finite number of hyperplanes. Two functions $f, g: \mathbb{Z}^k \rightarrow K$ are *equal almost everywhere* (written $f = g$ a.e.) if there exists a set S of measure zero such that $f(\mathbf{z}) = g(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{Z}^k \setminus S$. A region $\mathcal{R} \subset \mathbb{Z}^k$ is *polyhedral* if $\mathcal{R} = \mathbb{Z}^k$ or \mathcal{R} is the intersection of a finite number of half-spaces.

It is easily seen (Lemma 3.2.4) that a function f is a hypergeometric term on \mathbb{Z}^k if and only if, for all $\mathbf{v} \in \mathbb{Z}^k$, there exist nonzero polynomials $A_{\mathbf{v}}$ and $B_{\mathbf{v}} \in K[\mathbf{z}]$ such that $A_{\mathbf{v}} f = B_{\mathbf{v}} f^{\mathbf{v}}$ a.e. A hypergeometric term f on \mathbb{Z}^k is *honest* if for all $\mathbf{v} \in \mathbb{Z}^k$ there exist *relatively prime* polynomials $A_{\mathbf{v}}$ and $B_{\mathbf{v}} \in K[\mathbf{z}]$ such that $A_{\mathbf{v}} f = B_{\mathbf{v}} f^{\mathbf{v}}$ a.e.

The main results of Chapter 3 are Theorem 3.7.1 and its corollary for algebraically closed fields, Corollary 3.7.3.

Theorem 3.7.1 Let f be an honest hypergeometric term on \mathbb{Z}^k . There exist relatively prime polynomials C and $D \in K[z]$, a finite set $V \subset \mathbb{Z}^k$, univariate polynomials $a_v, b_v \in K[z]$, for each $v \in V$, and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_m$ such that

- (1) \mathbb{Z}^k is the disjoint union of the \mathcal{R}_i and a set of measure zero;
- (2) for each $i \in \{1, \dots, m\}$ there exists $z_0 \in \mathcal{R}_i$ such that $C(z_0) \neq 0$, and for all $z \in \mathcal{R}_i$ for which $D(z) \neq 0$,

$$f(z) = f(z_0) \frac{C(z)}{C(z_0)} \frac{D(z_0)}{D(z)} \prod_{v \in V} \prod_j^{z \cdot v} \frac{a_v(j)}{b_v(j)}.$$

- (3) all the terms $a_v(j)$ and $b_v(j)$ occurring in the product are nonzero.

The hypergeometric terms that occur in practice can be expressed as products of Pochhammer symbols, so the question arises: Is this true in general? Corollary 3.7.3 show that if the field K is algebraically closed and the hypergeometric term is honest, then the answer is yes, at least piecewise.

Corollary 3.7.3 Let f be an honest hypergeometric term on \mathbb{Z}^k over a field K that is algebraically closed. Then there exist relatively prime polynomials C and $D \in K[z]$ and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\gamma \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that

- (1) for all $z \in \mathcal{R}_\ell$ such that $D(z) \neq 0$,

$$f(z) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} \frac{C(z)}{D(z)} \frac{\prod_{i=1}^p (m_i)_{v_i \cdot z + r_i}}{\prod_{j=1}^q (n_j)_{w_j \cdot z + s_j}};$$

- (2) for all i and j and all $z \in \mathcal{R}_\ell$, $v_i \cdot z + r_i$ and $w_j \cdot z + s_j$ are positive;
- (3) the Pochhammer symbols occurring in the products are nonzero.

If we assume that everything in sight is nonzero, then the expression for f in Theorem 3.7.1 follows by induction from the expression for the term ratios in

Theorem 2.8.4. Unfortunately, zeros of C , D and the a_v and b_v are deadly to the induction. The polyhedral regions \mathcal{R}_i arise as regions for which the nonzero points of these polynomials have a certain connectedness property.

1.4 A holonomic hypergeometric term is piecewise proper

We prove the following result which settles the discrete part of Wilf and Zeilberger's conjecture that a hypergeometric term is holonomic if and only if it is proper.

Theorem 4.1.10 A holonomic hypergeometric term f on \mathbb{Z}^k over an algebraically closed field K is piecewise proper; there exist a polynomial $C \in K[z]$ and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\gamma \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that

(1) for all $z \in \mathcal{R}_\ell$,

$$f(z) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} C(z) \frac{\prod_{i=1}^p (m_i)_{v_i \cdot z + r_i}}{\prod_{j=1}^q (n_j)_{w_j \cdot z + s_j}};$$

(2) for all i and j and all $z \in \mathcal{R}_\ell$, $v_i \cdot z + r_i$ and $w_j \cdot z + s_j$ are positive;

(3) the Pochhammer symbols occurring in the products are nonzero.

Theorem 4.1.10 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

The proof of Theorem 4.1.10 uses the results of Chapter 3 and only elementary facts about holonomic functions that can be found in [16] and [3].

1.5 A Solution to a Problem of Cameron On Sum-free Complete Sets

For any subsets A and B of an additive group G , define $A + B = \{a + b : a \in A \text{ and } b \in B\}$ and $-A = \{-a : a \in A\}$. A subset S of G is said to be sum-free, complete, and symmetric respectively if $S + S \subset S^c$, $S + S \supset S^c$, and $S = -S$. Hence, S is sum-free and complete if and only if $S + S = S^c$.

Cameron observed that for any sufficiently small modulus m , every sum-free complete set in $\mathbb{Z}/m\mathbb{Z}$ is also symmetric. In fact, Calkin found that $m = 36$ is the smallest modulus for which there is a sum-free complete set that is not symmetric [7]. Cameron asked if there exists such a set for all sufficiently large moduli [7]. We answer Cameron's question by showing there exists such a set for all moduli greater than or equal to 890626.

We also show that every sum-free complete set in $\mathbb{Z}/m\mathbb{Z}$ that is not symmetric can be used to construct a counterexample to a conjecture of J.H. Conway. Conway conjectured that for any finite set S of integers, $|S + S| \leq |S - S|$. Conway's conjecture was disproved by Marica [9]. Later Stein showed how to make the ratio $|S + S|/|S - S|$ arbitrarily large [12]. We show that if S is sum-free and complete modulo m but not symmetric, then $|S + S| > |S - S|$; hence, S is a counterexample to a modular version of Conway's conjecture. Further, we show that if $S' \subset \mathbb{Z}$ is a certain set derived from S , then $|S' + S'| > |S' - S'|$; hence, S' is a counterexample to Conway's conjecture proper.

The history of sum-free sets begins with Schur who showed that the positive integers can not be partitioned into finitely many sum-free sets [11]. Sum-free sets have been used to find lower bounds for Ramsey numbers (see pp. 28, 128, 264 in [13]). Cameron describes some applications of sum-free sets and poses several problems [5],[6],[7]. George Andrews observed that sum-free complete sets play a role in partition identities (personal communication). For example, the set $\{1, 4\} \in$

$\mathbb{Z}/5\mathbb{Z}$, which arises in the Rogers-Ramanujan Identities (p. 109 in [1]), is sum-free, complete, and symmetric. Calkin showed that the number of sum-free sets contained within the first n integers is $o(2^{n(1/2+\epsilon)})$ for every $\epsilon > 0$ [4].

Chapter 2

The Rational Solutions of

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

The main result of this chapter is Theorem 2.8.4 [Ore–Sato], which reveals the multiplicative structure of solutions to the relation for the term ratios.

Theorem 2.8.4 Let $R_i \in K(\mathbf{z})$, $i = 1, \dots, k$, be rational functions such that

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. Then there exist polynomials C and $D \in K[\mathbf{z}]$, a finite set $V \subset \mathbb{Z}^k$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ for each $\mathbf{v} \in V$ such that for all $i \in \{1, \dots, k\}$,

$$R_i(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{e}_i)}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{e}_i)} \prod_{\mathbf{v} \in V} \prod_j^{v_i} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}.$$

Recall that the symbol \prod_a^b denotes $\prod_{j=a}^{b-1}$ if $b > a$, 1 if $a = b$, and $1/\prod_{j=b}^{a-1}$ if $a > b$.

We require the following definitions.

Definition 2.0.1 Define the *lead term* of a nonzero polynomial $p \in K[z_1, \dots, z_k]$ to be the nonzero term $c z_1^{n_1} \cdots z_k^{n_k}$ of p for which the degree vector (n_1, \dots, n_k) is maximal in the standard ordering. The polynomial p is *monic* if the coefficient of the lead term is 1.

For any $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in (K(\mathbf{z}))^k$ such that $\gamma_i \neq 0$ for each $i \in \{1, \dots, k\}$ and $\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{Z}^k$, we denote $\gamma_1^{v_1} \cdots \gamma_k^{v_k}$ by $\boldsymbol{\gamma}^{\mathbf{v}}$.

Lemma 2.0.2 Let $\mathbf{z} = (z_1, \dots, z_k)$, let $\mathbf{m} \in \mathbb{Z}^k$, and let $p = \sum_{\mathbf{n} \in \mathbb{Z}^k} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in K[z_1, \dots, z_k]$ be a polynomial. If the term $c_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ is maximal in the sense that

$z^m \mid z^n$ and $m \neq n$ imply $c_n = 0$, then for any $v \in \mathbb{Q}^k$, the coefficient of z^m in p^v is equal to the coefficient c_m of z^m in p .

Proof. By the binomial theorem, if z^m is not a proper divisor of z^n , then the coefficient of z^m in $(z + v)^n - z^n$ is 0. If z^m is a proper divisor of z^n , then $c_n = 0$. Hence, in either case, the coefficient of z^m in $c_n((z + v)^n - z^n)$ is 0, and, hence, the coefficient of z^m in $p^v - p = \sum c_n((z + v)^n - z^n)$ is 0. \square

Corollary 2.0.3 *If $p \in K[z_1, \dots, z_k]$ is monic, then so is p^v for any $v \in \mathbb{Q}^k$.*

Corollary 2.0.4 *If $p \in K(z_1, \dots, z_k)$ is nonzero, $v \in \mathbb{Q}^k$, $c \in K$, and $p^v = cp$, then $c = 1$.*

2.1 Gosper's lemma

We require the following adaptation of Petkovšek's refinement [10] of Gosper's lemma [8].

Lemma 2.1.1 *Let K be a field of characteristic 0, and let $R \in K(z_1, \dots, z_k)$ be a nonzero rational function. There exist polynomials A, B, C , and $D \in K[z_1, \dots, z_k]$ such that*

$$R = \frac{A}{B} \frac{C^{e_1}}{C} \frac{D}{D^{e_1}}$$

and

- (1) A and B^{ne_1} are relatively prime for all $n \in \mathbb{Z}$,
- (2) A and CD^{e_1} are relatively prime,
- (3) B and $C^{e_1}D$ are relatively prime, and
- (4) C and D are relatively prime.

Proof. The lemma is proved by a double application of Gosper's lemma. The gcds below are with respect to z_1 . By Gosper's lemma there exist a, b , and $c \in K(z_2, \dots, z_k)[z_1]$ such that $R = \frac{a}{b} \frac{c^{e_1}}{c}$ where $\gcd(a, b^{ne_1}) = 1$ for all nonnegative integers n and by Petkovšek's refinement we may assume in addition that $\gcd(a, c) =$

1 and $\gcd(b, c^{e_1}) = 1$. Applying this principle again, this time to $\frac{b}{a}$, we can write

$$(1) \quad \frac{b}{a} = \frac{B}{A} \frac{d^{e_1}}{d}$$

where $A, B, d \in K(z_2, \dots, z_k)[z_1]$, $\gcd(B, A^{ne_1}) = 1$ for all integers $n \geq 0$, $\gcd(B, d) = 1$, and $\gcd(A, d^{e_1}) = 1$. Since $\gcd(B, Ad) = 1$, it follows from (1) that $B \mid b$. Similarly, $A \mid a$. Hence, $\gcd(A, B^{ne_1}) = 1$ for all nonnegative integers n . On the other hand, since $\gcd(B, A^{ne_1}) = 1$ for all nonnegative integers n , it follows that $\gcd(A, B^{ne_1}) = 1$ for all nonpositive integers n . Hence $\gcd(A, B^{ne_1}) = 1$ for all integers n , positive, negative, or zero. Let $g = \gcd(c, d)$, let $C = c/g$, and let $D = d/g$. It's easily seen that $R = \frac{A}{B} \frac{C^{e_1}}{C} \frac{D}{D^{e_1}}$, $\gcd(A, CD^{e_1}) = 1$, $\gcd(B, C^{e_1}) = 1$, and $\gcd(C, D) = 1$.

For some α and $\beta \in K[z_2, \dots, z_k]$, αA and βB are in $K[z]$. Replace A with $\alpha\beta A$ and B with $\alpha\beta B$; the ratio A/B is unchanged, hence, R is unchanged. Similarly, adjust C and D so that A, B, C , and D are in $K[z]$. If A and B have a common factor d (which must be free of z_1), we replace A and B by A/d and B/d . Since $d = d^{ne_1}$ it follows that A and B^{ne_1} are relatively prime for $n \in \mathbb{Z}$. If C has any divisor d that is free of z_1 , replace C with C/d . Since $d^{e_1} = d$, C^{e_1}/C is unchanged. Similarly, adjust D . It follows that A_1 and CD^{e_1} are relatively prime, B_1 and $C^{e_1}D$ are relatively prime, C and D are relatively prime, and $R = \frac{A}{B} \frac{C^{e_1}}{C} \frac{D}{D^{e_1}}$. \square

Definition 2.1.2 Let $R \in K(z_1, \dots, z_k)$ be a rational function. The rational function R is said to be *simple* over K if it can be written in the form

$$R(\mathbf{z}) = \bar{R}(\mathbf{v} \cdot \mathbf{z}) = \bar{R}(v_1 z_1 + \dots + v_k z_k)$$

where $\bar{R} \in K(z)$ is a univariate rational function and $\mathbf{v} \in \mathbb{Q}^k$.

Hence, $(z_1^2 + z_2 + 3z_3)^2$ is simple over $\mathbb{Q}(z_1)$ but (apparently) not over \mathbb{Q} . It's easily proved that it's not simple over \mathbb{Q} using Corollary 2.5.4.

Lemma 2.1.3 *Let K be a field of characteristic 0, and let $R_1, \dots, R_k \in K(z_1, \dots, z_k)$ be nonzero rational functions satisfying the relation*

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. Then there exist polynomials $A_1, \dots, A_k, B_1, \dots, B_k, C$, and D such that

$$R_i = \frac{A_i}{B_i} \frac{C^{e_i}}{C} \frac{D}{D^{e_i}}$$

for $i \in \{1, \dots, k\}$, and the irreducible divisors of A_i and B_i are all simple for $i \in \{1, \dots, k\}$.

Proof. The one-variable case of Lemma 2.1.3 is trivial. The proof of the two-variable case continues through section 2.4 and the proof of the higher dimensional cases occupies sections 2.5 through 2.7.

We prove the two-variable case. In fact, we prove a stronger version of the two-variable case that is needed for the proof of the higher dimensional cases: if

$$R_1 = \frac{\bar{A}_1}{\bar{B}_1} \frac{\bar{C}^{e_1}}{\bar{C}} \frac{\bar{D}}{\bar{D}^{e_1}}$$

and \bar{A}_1 and $\bar{B}_1^{ne_1}$ are relatively prime for all $n \in \mathbb{Z}$, then we may take $A_1 = \bar{A}_1$, $B_1 = \bar{B}_1$, $C = \bar{C}$, and $D = \bar{D}$. By Lemma 2.1.1, such \bar{A}_1 , \bar{B}_1 , \bar{C} , and \bar{D} exist. Thus, we assume that

$$R_1 = \frac{A_1}{B_1} \frac{C^{e_1}}{C} \frac{D}{D^{e_1}}$$

where

$$(1) \quad A_1 \text{ and } B_1^{ne_1} \text{ are relatively prime for all } n \in \mathbb{Z}.$$

Let $r_i = R_i \frac{C}{C^{e_i}} \frac{D^{e_i}}{D}$. Thus, $r_1 = \frac{A_1}{B_1}$. Let A_2 and B_2 be relatively prime polynomials such that $r_2 = \frac{A_2}{B_2}$. Using $R_1 R_2^{e_1} = R_2 R_1^{e_1}$ and the definition of r_i , it's easily verified that $r_1 r_2^{e_1} = r_2 r_1^{e_2}$ and, hence,

$$(2) \quad A_1 A_2^{e_1} B_1^{e_2} B_2 = A_1^{e_2} A_2 B_1 B_2^{e_1}.$$

We will show that the irreducible divisors of the left side of (2) are simple.

2.2 The divisibility lattice path

For any irreducible divisor d we construct a lattice path $\{T_i\}$ in \mathbb{Z}^2 such that for all $i \geq 0$, d^{T_i} divides one of the four factors on the left side of (2). Further, the factor T_i divides is determined by the value of $S_i = T_i - T_{i-1}$, which, since T_i is a lattice path, is one of the four directions \mathbf{e}_1 , $-\mathbf{e}_1$, \mathbf{e}_2 , and $-\mathbf{e}_2$.

To be precise, define

$$X(-\mathbf{e}_2) = A_1, \quad X(\mathbf{e}_1) = A_2^{\mathbf{e}_1}, \quad X(\mathbf{e}_2) = B_1^{\mathbf{e}_2}, \quad X(-\mathbf{e}_1) = B_2.$$

We will construct T_i so that

$$(3) \quad d^{T_i} \mid X(S_i), \quad \text{where } S_i = T_i - T_{i-1}.$$

The product of the values of X is the left side of (2). It will be useful to also define

$$Y(-\mathbf{e}_2) = A_2^{\mathbf{e}_1}, \quad Y(\mathbf{e}_1) = A_2, \quad Y(\mathbf{e}_2) = B_1, \quad Y(-\mathbf{e}_1) = B_2^{\mathbf{e}_1}$$

so that the product of the values of Y is the right side of (2). The relationship between terms on the left side of (2) and terms on the right side is captured by the shift relation

$$(4) \quad (Y(\mathbf{v}))^{\mathbf{v}} = X(\mathbf{v})$$

and coprimeness condition

$$(5) \quad X(\mathbf{v}) \text{ and } Y(-\mathbf{v}) \text{ are relatively prime}$$

for $\mathbf{v} \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$, which are easily verified.

We construct the lattice path $\{T_i\}$ by induction. Let an irreducible divisor d of the left side of (2) be given. Since d is irreducible, d must divide $X(\mathbf{v})$ for some $\mathbf{v} \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. Define $T_0 = 0$ and $S_0 = \mathbf{v}$. Thus, $d^{T_0} \mid X(S_0)$. Now assume inductively that T_i and S_i have been defined for $0 \leq i \leq j$. Since d^{T_j} divides

the left side of (2), it must also divide the right side, and since d^{T_j} is irreducible, $d^{T_j} \mid Y(\mathbf{v})$ for some $\mathbf{v} \in \{\pm \mathbf{e}_1, \pm \mathbf{e}_2\}$. Let $S_{j+1} = \mathbf{v}$ and let $T_{j+1} = T_j + S_{j+1}$. Since $d^{T_j} \mid Y(S_{j+1})$, it follows that $d^{T_j + S_{j+1}} \mid (Y(S_{j+1}))^{S_{j+1}}$, and hence $d^{T_{j+1}} \mid X(S_{j+1})$ by (4). This completes the induction step. We have constructed T_i and S_i satisfying (3).

2.3 What goes up must not come down

Having constructed the path T , we explore what restrictions the coprimeness conditions (5) and (1) impose on the shape of T .

From the condition (5) we deduce path T does not traverse the same segment of the lattice path in opposite directions. By the path T traversing a segment of the lattice in the direction \mathbf{v} , we mean for some $i \geq 1$, the end points of the segment are T_{i-1} and T_i , and $T_i - T_{i-1} = \mathbf{v}$.

Suppose to the contrary the path traverses the same segment in opposite directions. Then for some i and $j > 0$ we have $S_i = -S_j$ and $T_i = T_{j-1}$. By (3), $d^{T_j} \mid X(S_j)$, hence, by (4), $d^{T_{j-1}} \mid Y(S_j)$, hence $d^{T_i} \mid Y(S_j)$, and hence $d^{T_i} \mid Y(-S_i)$. But $d^{T_i} \mid X(S_i)$ which contradicts condition (5).

Taking $j = i + 1$ we see that the path can't reverse directions without going in a perpendicular direction first ($S_i \neq -S_{i+1}$).

From the condition (1) we deduce that the path T doesn't go up a segment and come down one of its horizontal translates. (By go up and come down, we mean traverse in the directions \mathbf{e}_2 and $-\mathbf{e}_2$, respectively.) Suppose the contrary. Then for some i and $j > 0$ and $n \in \mathbb{Z}$ we have $S_i = -\mathbf{e}_2$, $S_j = \mathbf{e}_2$, and $T_i = T_{j-1} + n\mathbf{e}_1$. By (3), $d^{T_j} \mid X(S_j)$, hence by (4) $d^{T_{j-1}} \mid Y(S_j)$, hence $d^{T_i - n\mathbf{e}_1} \mid Y(S_j)$, hence by the definition of Y , $d^{T_i - n\mathbf{e}_1} \mid B_1$, and hence $d^{T_i} \mid B_1^{n\mathbf{e}_1}$. But $d^{T_i} \mid X(S_i) = A_1$, which contradicts (1).

The condition that the path T doesn't go up a segment and come down one of

its horizontal translates immediately implies a stronger condition:

(6) If T goes up any segment, it must not come down any segment.

Suppose to the contrary T goes up to T_i and down to T_j ($S_i = \mathbf{e}_2$ and $S_j = -\mathbf{e}_2$), and assume that $|i - j|$ is minimal. By the minimality of $|i - j|$, $S_k \neq \pm \mathbf{e}_2$ for any k between i and j . It follows that the segments (T_{i-1}, T_i) and (T_{j-1}, T_j) are horizontal translates of each other, contradicting the original condition.

Finally, from (6) it's easy to deduce that the path T is unbounded. If the path is unbounded vertically, we are done. Thus, we may assume that for some $k \geq 0$, $|T_i \cdot \mathbf{e}_2| \leq k$ for all i . It follows from (6) that the path can go in a vertical direction at most k times. Since the path can't reverse direction without going in a perpendicular direction first, the path can change horizontal direction at most k times. Hence the path is unbounded horizontally.

2.4 The divisor d is simple.

Using the fact that the path T is unbounded, we show that it satisfies a nontrivial relation $d = d^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}^2$. By the fact that T is unbounded, the set $\mathcal{T} = \{T_i : i \geq 0\}$ of values assumed by T is infinite. Let P be the left side of (2). For every $t \in \mathcal{T}$, $d^t \mid P$. Since \mathcal{T} is infinite and P has only finitely many monic divisors, $cd^{t_1} = d^{t_2}$ for some t_1 and $t_2 \in \mathcal{T}$ and $c \in K$. By Corollary 2.0.4, $c = 1$. Letting $\mathbf{v} = t_1 - t_2$, we have $d^{\mathbf{v}} = d$.

Using the equation $d = d^{\mathbf{v}}$, we show that d is the composition of a univariate polynomial and a linear polynomial:

$$(7) \quad d(z_1, z_2) = p(v_2 z_1 - v_1 z_2) \quad \text{where } p \in K[z].$$

We may assume without loss of generality that $v_2 \neq 0$. Define

$$q(z_1, z_2) = d(z_1 + v_1 z_2, v_2 z_2).$$

Thus, $q = q^{e^2}$. Iterating the last equation, $q = q^{ne^2}$ for all positive integers n . Thus, $q(z_1, z_2) = q(z_1, z_2 + n)$ identically. The left side of the last equation is free of n and the right side is symmetric in n and z_2 ; thus, the left side must also be free of z_2 . Hence q is free of its second argument. It's easily verified that $d(z_1, z_2) = q(z_1 - (v_1/v_2)z_2, z_2/v_2)$. Letting $p(z) = q(z/v_2, 0)$, (7) is proved and, hence, d is simple.

This completes the proof of the two-variable case.

2.5 The rational Galois space of a rational function

We require the following definitions and lemmas for the proof of the $k > 2$ case of Lemma 2.1.3.

Definition 2.5.1 Let $S = \{x_1, \dots, x_k\}$ be a finite set that is algebraically independent over K , and let $R \in K(S)$ be a rational function in $\mathbf{x} = (x_1, \dots, x_k)$. We define the *rational Galois space* of R over K with respect to \mathbf{x} to be the set

$$\text{rgal}(R, \mathbf{x}, K) = \{\mathbf{v} \in \mathbb{Q}^k : R(\mathbf{x} + \mathbf{v}) = R(\mathbf{x})\}.$$

We write $\text{rgal}(R)$ for $\text{rgal}(R, \mathbf{z}, K)$; thus, $\text{rgal}(R) = \{\mathbf{v} \in \mathbb{Q}^k : R^{\mathbf{v}} = R\}$.

Let G be the Galois group of $K(\mathbf{x})$ over K . It's easily seen that $\text{rgal}(R, \mathbf{x}, K)$ is isomorphic to the subgroup $\{g \in G : g(R) = R \text{ and } g(x_i) - x_i \in \mathbb{Q} \text{ for } i \in \{1, \dots, k\}\}$ of G .

We show that the rational Galois space is a subspace of \mathbb{Q}^k . It's clearly closed under vector addition. By iterating $R^{\mathbf{v}} = R$, it follows that $R^{n\mathbf{v}} = R$ for any positive integer n . Writing $R = A/B$ where A and B are polynomials, it follows that $AB^{n\mathbf{v}} = BA^{n\mathbf{v}}$ for all positive integers n . Each side of the last equation is a polynomial in n . Thus, $AB^{n\mathbf{v}} = BA^{n\mathbf{v}}$ identically. Hence $R^{n\mathbf{v}} = R$ for all $n \in \mathbb{Q}$ and the rational Galois space is closed under multiplication by scalars.

Lemma 2.5.2 Let $R \in K(z_1, \dots, z_k)$ be a rational function such that $e_j \in \text{rgal}(R)$. Then R is free of z_j .

Proof. Since $R = R^{ne_j}$ for all $n \in \mathbb{Q}$, it follows that

$$R(z_1, \dots, z_k) = R(z_1, \dots, z_j + n, \dots, z_k)$$

identically. The left side is free of n and the right side is symmetric in z_j and n , so the left side must be free of z_j . \square

Lemma 2.5.3 *Let $R \in K(z_1, \dots, z_k)$ be a rational function and let M be an $r \times k$ matrix over \mathbb{Q} . If the kernel of M is equal to the rational Galois space of R , then there exists a rational function $\bar{R} \in K(z_1, \dots, z_r)$ such that $R(\mathbf{z}) = \bar{R}(M\mathbf{z})$.*

Proof. Let A and B be invertible $k \times k$ and $r \times r$ matrices respectively such that the $r \times k$ matrix AMB is of the form

$$AMB = \begin{bmatrix} I_{j,j} & Z_{j,k-j} \\ Z_{r-j,j} & Z_{r-j,k-j} \end{bmatrix}$$

where j is the rank of M , $I_{j,j}$ is the $j \times j$ identity matrix, and $Z_{\ell,m}$ is the $\ell \times m$ zero matrix. Since j is the rank of M , it follows that $j \leq r$. Let $R_1(\mathbf{z}) = R(B\mathbf{z})$. Then $\text{rgal } R_1 = B^{-1} \text{rgal } R = B^{-1} \ker M = \ker MB = \ker AMB = \text{span}(\mathbf{e}_{j+1}, \dots, \mathbf{e}_k)$. By Lemma 2.5.2, $R_1(\mathbf{z})$ is free of z_{j+1}, \dots, z_k . It follows that $R_1(\mathbf{z}) = \bar{R}_1(AMB\mathbf{z})$, where $\bar{R}_1 \in K(z_1, \dots, z_r)$ is defined by

$$\bar{R}_1(z_1, \dots, z_r) = R_1(z_1, \dots, z_k) = R_1(z_1, \dots, z_j, 0, \dots, 0).$$

Letting $\bar{R}(\mathbf{z}) = \bar{R}_1(A\mathbf{z})$, we have $R_1(\mathbf{z}) = \bar{R}(MB\mathbf{z})$. Hence $R(\mathbf{z}) = R_1(B^{-1}\mathbf{z}) = \bar{R}(M\mathbf{z})$ as claimed. \square

Corollary 2.5.4 *A nontrivial rational function $R \in K(z_1, \dots, z_k)$ is simple if and only if the dimension of $\text{rgal } R$ is $k - 1$.*

Proof. If R is simple, then there exists $\mathbf{v} \in \mathbb{Q}^k$ and a univariate rational function $r \in K(\mathbf{z})$ such that $R(\mathbf{z}) = r(\mathbf{z} \cdot \mathbf{v})$. Clearly $\text{rgal } R$ contains the space orthogonal to \mathbf{v} , and thus $\dim \text{rgal } R \geq k - 1$. If the dimension were k , then R would be trivial; therefore, the dimension must be $k - 1$.

Conversely, if the dimension of $\text{rgal}(R)$ is $k - 1$, there exists a $1 \times k$ matrix $\mathbf{v} = [v_1, \dots, v_k]$ with kernel equal to $\text{rgal } R$. By Lemma 2.5.3, $R(\mathbf{z}) = \bar{R}(\mathbf{v} \cdot \mathbf{z}) = \bar{R}(v_1 z_1 + \dots + v_k z_k)$, so R is simple. \square

Lemma 2.5.5 *Let d and $A \in K[z_1, \dots, z_k]$ be polynomials. If $d \mid A$, then $\text{rgal } d \supseteq \text{rgal } A$.*

Proof. Suppose the contrary. Then the quotient space $\text{rgal } A / (\text{rgal } A \cap \text{rgal } d)$ is infinite. Let S be a set of representatives of the cosets.

For any $s \in S$, $A^{-s} = A$, hence $d \mid A^{-s}$, hence $d^s \mid A$. Since S is infinite and A has only finitely many monic divisors, $d^{s_1} = c d^{s_2}$ for some s_1 and $s_2 \in S$ and $c \in K$. Letting $s = s_1 - s_2$, $d^s = c d$ and, by a corollary of Lemma 2.0.2, $c = 1$. But $d^{s_1} = d^{s_2}$, contradicting the definition of s . \square

Corollary 2.5.6 *If A and $B \in K[z_1, \dots, z_k]$ are polynomials such that $\text{rgal } A = \text{rgal } B = s$, then $\text{rgal } AB = s$.*

Proof. By Lemma 2.5.5, $\text{rgal } AB \subseteq s$. But clearly $\text{rgal } AB \supseteq s$, so $\text{rgal } AB = s$. \square

Lemma 2.5.7 *If A and $B \in K[z_1, \dots, z_k]$ are relatively prime polynomials, then*

$$\text{rgal } \frac{A}{B} \subseteq \text{rgal } A \cap \text{rgal } B.$$

Proof. Suppose the contrary. Then the quotient space

$$\text{rgal } \frac{A}{B} / (\text{rgal } \frac{A}{B} \cap \text{rgal } A \cap \text{rgal } B)$$

is infinite. Let S be a set of representatives of the cosets, and let s_1 and $s_2 \in S$.

Since $(A/B)^{s_1} = (A/B)^{s_2}$, letting $s = s_1 - s_2$ it follows that $(A^s/B^s) = (A/B)$. Hence $AB^s = BA^s$. Since A and B are relatively prime, $A \mid A^s$. Similarly, $A^s \mid A$. Hence $A = cA^s$, and hence, by corollary 2.0.4 of Lemma 2.0.2, $c = 1$. Hence $A = A^s$. Similarly, $B = B^s$. Hence $s_1 - s_2 = s \in \text{rgal}(A/B) \cap \text{rgal } A \cap \text{rgal } B$, contradicting the definition of s_1 and s_2 . \square

Lemma 2.5.8 Every rational function $R \in K(z_1, \dots, z_k)$ can be written as a product

$$R = g \prod_{s \in S} \frac{A_s}{B_s},$$

where S is the collection of all proper subspaces of \mathbb{Q}^k , $g \in K$, and, for all $s \in S$, A_s and $B_s \in K[z_1, \dots, z_k]$ are relatively prime monic polynomials such that if d is a nontrivial divisor of A_s or B_s , then $\text{rgal}(d) = s$. This expression is unique.

Of course, $A_s = B_s = 1$ for all but a finite number of $s \in S$.

Proof. We first show that such a factorization exists. Let $R = gA/B$, where A and B are relatively prime monic polynomials and $g \in K$. Write A and B as products of nontrivial irreducible monic factors

$$A = a_1 a_2 \cdots a_m \text{ and } B = b_1 \cdots b_n, \quad m, n \geq 0.$$

For all $s \in S$ let A_s be the product of all a_i such that $\text{rgal}(a_i) = s$, with the understanding that empty products are 1. Similarly define B_s . Clearly $A = \prod_{s \in S} A_s$ and $B = \prod_{s \in S} B_s$. Hence

$$R = g \prod_{s \in S} \frac{A_s}{B_s}$$

and A_s and B_s are relatively prime.

Let d be a nontrivial monic divisor of A_s or B_s . Then d can be expressed as a product of a subset of the a_i for which $\text{rgal } a_i = s$ and the b_i for which $\text{rgal } b_i = s$. Hence, by the corollary of Lemma 2.5.5, $\text{rgal } d = s$.

We show that the factorization is unique. Let

$$(1) \quad R = g \prod_{s \in S} \frac{A_s}{B_s} \text{ and } R = h \prod_{s \in S} \frac{C_s}{D_s}$$

be two factorizations satisfying the conditions of the lemma. We will show that $g = h$ and $A_s = C_s$ and $B_s = D_s$ for all $s \in S$. By (1),

$$(2) \quad g \prod A_s \prod D_s = h \prod B_s \prod C_s.$$

For any $t \in S$, A_t is relatively prime to $\prod_{s \neq t} B_s \prod_{s \neq t} C_s$ since any nontrivial irreducible divisor d of the former has $\text{rgal}(d) = s$ and any nontrivial divisor d of the latter has $\text{rgal}(d) \neq s$. Since A_t and B_t are relatively prime by assumption, it follows that $A_t \mid C_t$. Similarly $C_t \mid A_t$. Hence $C_t = A_t$. Similarly $B_t = D_t$. It follows that $g = h$. \square

2.6 Fixed factors

The complementary idea to the rational Galois space is the fixed factor.

Definition 2.6.1 Using Lemma 2.5.8, we define the *fixed factor* $\text{fix}_s(R)$ of a rational function $R \in K(z_1, \dots, z_k)$ for a subspace s of \mathbb{Q}^k . Let S be the set of all proper subspaces of \mathbb{Q}^k . Let $R = g \prod_{s \in S} (A_s/B_s)$ be the unique expression guaranteed by Lemma 2.5.8. Define

$$\text{fix}_s(R) = \begin{cases} \frac{A_s}{B_s} & \text{for all } s \in S \\ g & \text{for } s = \mathbb{Q}^k. \end{cases}$$

Lemma 2.6.2 Let $R \in K(z_1, \dots, z_k)$ be a rational function and let s be a subspace of \mathbb{Q}^k . Then

$$\text{rgal } \text{fix}_s R = s$$

unless $\text{fix}_s R = 1$.

Proof. If $\text{fix}_s R \neq 1$ then either $\text{fix}_s R$ has a nontrivial divisor or $\text{fix}_s R = c$ for some $c \in K$, $c \neq 1$. In the first case, $\text{rgal } \text{fix}_s R \subseteq \text{rgal } d = s$ by Lemma 2.5.5. But clearly $\text{rgal } \text{fix}_s R \supseteq s$, so $\text{rgal } \text{fix}_s R = s$. In the second case, $s = \mathbb{Q}^k$ and $\text{rgal } c = \mathbb{Q}^k$. \square

Lemma 2.6.3 For any nonzero rational functions R_1 and $R_2 \in K(z_1, \dots, z_k)$, any subspace s of \mathbb{Q}^k , and any $v \in \mathbb{Q}^k$

$$(1) \quad \text{fix}_s R_1 R_2 = \text{fix}_s R_1 \text{fix}_s R_2$$

$$(2) \quad \text{fix}_s \frac{1}{R_1} = \frac{1}{\text{fix}_s R_1}$$

$$(3) \quad \text{fix}_s R_1^v = (\text{fix}_s R_1)^v$$

Proof. Let S be the set of proper subspaces of \mathbb{Q}^k . Let $R_3 = R_1 R_2$, $R_4 = 1/R_1$, and $R_5 = R_1^v$. For $i = 1, 2, 3, 4$, and 5 let

$$R_i = g_i \prod_{s \in S} \frac{A_{i,s}}{B_{i,s}}$$

be the unique expression guaranteed by Lemma 2.5.8.

(1): Since $R_3 = R_1 R_2$, it follows that

$$g_3 \prod_{s \in S} \frac{A_{3,s}}{B_{3,s}} = g_1 g_2 \prod_{s \in S} \frac{A_{1,s} A_{2,s}}{B_{1,s} B_{2,s}}.$$

Let C_s and D_s be relatively prime polynomials such that

$$\frac{C_s}{D_s} = \frac{A_{1,s} A_{2,s}}{B_{1,s} B_{2,s}}.$$

Since $C_s \mid A_{1,s} A_{2,s}$ and every nontrivial divisor of $A_{1,s}$ or $A_{2,s}$ has $\text{rgal } d = s$, it follows by the corollary of Lemma 2.5.5 that every nontrivial divisor d of C_s has $\text{rgal } d = s$. Similarly, every nontrivial divisor d of D_s has $\text{rgal } d = s$. It follows that

$$R_3 = g_1 g_2 \prod_{s \in S} \frac{C_s}{D_s}$$

is the unique expression guaranteed by Lemma 2.5.8, hence

$$\frac{A_{3,s}}{B_{3,s}} = \frac{C_s}{D_s} = \frac{A_{1,s}}{B_{1,s}} \cdot \frac{A_{2,s}}{B_{2,s}}$$

for all $s \in S$. Hence $\text{fix}_s R_3 = \text{fix}_s R_1 \text{fix}_s R_2$ for all $s \in S$. Clearly $g_3 = g_1 g_2$ since the polynomials in (1) are monic, so $\text{fix}_s R_3 = \text{fix}_s R_1 \text{fix}_s R_2$ for $s = \mathbb{Q}^k$.

(2): We have

$$g_4 \prod_{s \in S} \frac{A_{4,s}}{B_{4,s}} = R_4 = \frac{1}{R_1} = \frac{1}{g_1} \prod_{s \in S} \frac{B_{1,s}}{A_{1,s}}.$$

It follows immediately that the factorization

$$R_4 = \frac{1}{g_1} \prod_{s \in S} \frac{B_{1,s}}{A_{1,s}}$$

satisfies the conditions of the lemma, hence $g_4 = 1/g_1$ and

$$\frac{A_{4,s}}{B_{4,s}} = \frac{B_{1,s}}{A_{1,s}}.$$

Hence $\text{fix}_s R_4 = 1/(\text{fix}_s R_1)$ for all $s \in S$ and for $s = \mathbb{Q}^k$.

(3): We have

$$g_5 \prod_{s \in S} \frac{A_{5,s}}{B_{5,s}} = R_5 = R_1^v = g_1^v \prod_{s \in S} \frac{A_{1,s}^v}{B_{1,s}^v}.$$

Since $\text{rgal } d^v = \text{rgal } d$ for any polynomial d and any $v \in \mathbb{Q}^k$,

$$R_5 = g_1^v \prod_{s \in S} \frac{A_{1,s}^v}{B_{1,s}^v}$$

satisfies the conditions of the lemma. Hence

$$g_5 = g_1^v \text{ and } \frac{A_{5,s}}{B_{5,s}} = \frac{A_{1,s}^v}{B_{1,s}^v}$$

for all $s \in S$. Hence $\text{fix}_s R_5 = (\text{fix}_s R_1)^v$ for all $s \in S$ and for $s = \mathbb{Q}^k$. \square

Lemma 2.6.4 *Let $R_1, \dots, R_k \in K(z_1, \dots, z_k)$ be rational functions satisfying the relation*

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. For $i = 1, \dots, k$ and all subspaces s of \mathbb{Q}^k define $R_{i,s} = \text{fix}_s(R_i)$. Then for all subspaces s of \mathbb{Q}^k and all $i, j \in \{1, \dots, k\}$,

$$R_{i,s} R_{j,s}^{e_i} = R_{j,s} R_{i,s}^{e_j}.$$

Proof. By Lemma 2.6.3,

$$\begin{aligned} \text{fix}_s R_i (\text{fix}_s R_j)^{e_i} &= \text{fix}_s R_i R_j^{e_i} \\ &= \text{fix}_s R_j R_i^{e_j} \\ &= \text{fix}_s R_j (\text{fix}_s R_i)^{e_j}. \end{aligned}$$

\square

Lemma 2.6.5 *Let $R_1, \dots, R_k \in K(z_1, \dots, z_k)$ be rational functions satisfying the relation*

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. Let s be a proper subspace of \mathbb{Q}^k such that $e_i \in s$. Then $\text{fix}_s R_i = 1$.

Proof. By Lemma 2.6.4, for $j \in \{1, \dots, k\}$,

$$\begin{aligned} \frac{\text{fix}_s R_i}{(\text{fix}_s R_i)^{e_j}} &= \frac{\text{fix}_s R_j}{(\text{fix}_s R_j)^{e_i}} \\ &= \frac{\text{fix}_s R_j}{\text{fix}_s R_j} \\ &= 1. \end{aligned}$$

Hence $e_j \in \text{rgal fix}_s R_i$. By Lemma 2.6.2, either $\text{fix}_s R_i = 1$ or $\text{rgal fix}_s R_i = s$. In the first case we are done. In the second case $e_j \in s$ for all $j \in \{1, \dots, k\}$. Hence $s = \mathbb{Q}^k$, contradicting the assumption that s is a proper subspace. \square

2.7 Proof of Lemma 2.1.3 for $k > 2$

Now we are in a position to prove the case $k > 2$ of Lemma 2.1.3. We proceed by induction. Assume the case of $k - 1$ variables is proved. Let

$$R_1 = \frac{A_1}{B_1} \frac{C_1}{C_1^{e_1}} \frac{D_1^{e_1}}{D_1}$$

as in Lemma 2.1.1. Let $K_{i,j} = K(\{z_1, \dots, z_k\} \setminus \{z_i, z_j\})$ and let $\mathbb{Q}_{i,j}$ be the subspace of \mathbb{Q}^k generated by e_i and e_j . Let $r_1 = A_1/B_1$.

Let $i \in \{2, \dots, k\}$. Applying the strong version of the two-variable case of Lemma 2.1.3 to R_1 and R_i over the field $K_{1,i}$, it follows that we can write

$$R_i = \frac{A_i}{B_i} \frac{C_1}{C_1^{e_i}} \frac{D_1^{e_i}}{D_1},$$

where the irreducible divisors of A_1 , A_i , B_1 , and B_i are all simple over $K_{1,i}$. The point of using the strong version of the two-variable case is that the same C_1 and D_1 can be used for all $i \in \{2, \dots, k\}$.

Let d be a nontrivial irreducible divisor of A_1 or B_1 . The subspace $\text{rgal}(d) \cap \mathbb{Q}_{1,j}$ of $\mathbb{Q}_{1,j}$ is isomorphic to the subspace $\text{rgal}(d, (z_1, z_j), K_{1,j})$ of \mathbb{Q}^2 . Hence $\text{rgal}(d) \cap \mathbb{Q}_{1,j}$ is a one-dimensional subspace of $\mathbb{Q}_{1,j}$ by Lemma 2.5.3. Let $s = \text{rgal}(d)$. Let $r_i = A_i/B_i$ for $i \in \{2, \dots, k\}$. It's easily verified that r_i satisfies

$$r_i r_j^{e_i} = r_j r_i^{e_j}$$

for $i, j \in \{1, \dots, k\}$, hence, by Lemma 2.6.5, if $e_1 \in s$ then $\text{fix}_s R_1 = 1$. But $\text{fix}_s R_1 = 1$ implies $\text{fix}_s A_1 = \text{fix}_s B_1 = 1$, which implies $d = 1$. Since d is nontrivial, $e_1 \notin s$.

Let v_i be a nonzero vector in $\text{rgal } d \cap \mathbb{Q}_{1,i}$. Since $e_1 \notin s$, it follows that $e_i \cdot v_i \neq 0$. Let $t = \sum_{i=2}^k c_i v_i$. It is easily seen that the vectors v_2, \dots, v_k are linearly independent: if $c_j \neq 0$, then $e_j \cdot t = c_j(e_j \cdot v_j)$ is nonzero, hence t is nonzero.

It follows that the dimension of $\text{rgal}(d)$ is at least $k - 1$. By Lemma 2.5.3, d is simple over K . We have shown all the nontrivial irreducible divisors of A_1 and B_1 are simple over K .

Define \bar{R}_i and \tilde{R}_i by

$$\bar{R}_i = \prod \text{fix}_s r_i$$

and

$$\tilde{R}_i = \prod' \text{fix}_s r_i$$

where the first product is taken over all subspaces s of \mathbb{Q}^k such that $e_1 \notin s$, and the second is taken over all subspaces such that $e_1 \in s$. Hence $r_i = \bar{R}_i \tilde{R}_i$. It follows from Lemma 2.6.4 that

$$\bar{R}_i \bar{R}_j^{e_i} = \bar{R}_j \bar{R}_i^{e_j}$$

and

$$\tilde{R}_i \tilde{R}_j^{e_i} = \tilde{R}_j \tilde{R}_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$.

Let \bar{A}_i and \bar{B}_i be relatively prime polynomials such that $\bar{R}_i = \bar{A}_i/\bar{B}_i$. We claim that the irreducible divisors of \bar{A}_i and \bar{B}_i are simple. By Lemma 2.6.5, $\tilde{R}_1 = 1$. Hence $\bar{R}_1 = r_1$, and hence the irreducible divisors of $\bar{A}_1 = A_1$ and $\bar{B}_1 = B_1$ are simple. Let d be an irreducible divisor of \bar{A}_i or \bar{B}_i for $i \in \{2, \dots, k\}$, and let $s = \text{rgal}(d)$. If the dimension of s is less than $k - 1$, then $\text{fix}_s \bar{R}_1 = 1$ since all the irreducible divisors of \bar{A}_1 and \bar{B}_1 are simple and, by Lemma 2.5.3, have rational Galois spaces of dimension k or $k - 1$. By Lemma 2.6.4,

$$\frac{\text{fix}_s \bar{R}_i}{\text{fix}_s \bar{R}_i^{e_1}} = \frac{\text{fix}_s \bar{R}_1}{\text{fix}_s \bar{R}_1^{e_1}} = 1,$$

so $e_1 \in \text{rgal}(\text{fix}_s \bar{R}_i)$. Hence $e_1 \in s$ by Lemma 2.6.2, which contradicts the definition of \bar{R}_i , so the dimension of $\text{rgal} d$ is $k - 1$ or k and d is simple by Lemma 2.5.3.

By Lemma 2.5.2, \tilde{R}_i is free of z_i for all $i \in \{1, \dots, k\}$. Inductively applying the $k - 1$ variable case of Lemma 2.1.3 to $\tilde{R}_2, \dots, \tilde{R}_k \in K(z_2, \dots, z_k)$, there exist \tilde{C} and \tilde{D} such that

$$(1) \quad \tilde{R}_i = \frac{\tilde{A}_i}{\tilde{B}_i} \frac{\tilde{C}}{\tilde{C}^{e_i}} \frac{\tilde{D}^{e_i}}{\tilde{D}}$$

for $i \in \{2, \dots, k\}$, and the irreducible divisors of \tilde{A}_i and \tilde{B}_i are simple over K . Let $\tilde{A}_1 = \tilde{B}_1 = 1$. Then (1) is true for $i = 1$.

Letting $C = C_1 \tilde{C}$, $D = D_1 \tilde{D}$, $a_i = \bar{A}_i \tilde{A}_i$, and $b_i = \bar{B}_i \tilde{B}_i$,

$$R_i = \frac{a_i}{b_i} \frac{C}{C^{e_i}} \frac{D^{e_i}}{D}$$

where the irreducible divisors of a_i and b_i are all simple. This completes the proof of Lemma 2.1.3. \square

2.8 The multiplicative structure of R_i

Lemma 2.8.1 *Let $R_1, \dots, R_k \in K(z_1, \dots, z_k)$ be simple rational functions with the same rational Galois space such that*

$$(1) \quad R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. Then there exist $\mathbf{v} \in \mathbb{Z}^k$, a univariate rational function $t \in K(z)$, and $c_1, \dots, c_k \in K$ such that

$$R_i(z_1, \dots, z_k) = c_i \prod_{j=0}^{v_i} t(\mathbf{v} \cdot \mathbf{z} + j)$$

for $i \in \{1, \dots, k\}$.

Proof. Let $s = \text{rgal } R_i$. The case $s = \mathbb{Q}^k$ is trivial: if $s = \mathbb{Q}^k$, then $R_i \in K$ for all $i \in \{1, \dots, k\}$, so we may take $c_i = R_i$, $t = 1$, and $\mathbf{v} = \mathbf{0}$.

We consider the case $s \neq \mathbb{Q}^k$. By the corollary of Lemma 2.5.3 the dimension of s is $k - 1$. Hence, there exists a vector $\mathbf{v} \in \mathbb{Z}^k$ with $\text{gcd}(\mathbf{v}) = 1$ such that the kernel of the $1 \times k$ matrix \mathbf{v} is s . By Lemma 2.5.3, for all $i \in \{1, \dots, k\}$ there exist univariate $r_i \in K(z)$ such that $R_i(\mathbf{z}) = r_i(\mathbf{v} \cdot \mathbf{z})$.

Since $\text{gcd}(\mathbf{v}) = 1$, there exists $\mathbf{w} \in \mathbb{Z}^k$ such that $\mathbf{w} \cdot \mathbf{v} = w_1 v_1 + \dots + w_k v_k = 1$. For any univariate rational function $r \in K(z)$ and any $n \in \mathbb{Z}$, the rational function $r^{(n)} \in K(z)$ is defined by $r^{(n)}(z) = r(z + n)$. Define

$$t = \prod_i^{w_1} r_1^{(v_1 i)} \prod_i^{w_2} r_2^{(v_1 w_1 + v_2 i)} \dots \prod_i^{w_k} r_k^{(v_1 w_1 + \dots + v_{k-1} w_{k-1} + v_k i)}.$$

We claim that $R_i(z_1, \dots, z_k) = \prod_j^{v_i} t^{(j)}(\mathbf{v} \cdot \mathbf{z})$. We first show that $r_i/r_i^{(1)} = t/t^{(v_i)}$.

Since

$$\begin{aligned} R_i^{e_j}(z_1, \dots, z_k) &= (r(\mathbf{v} \cdot \mathbf{z}))^{e_j} \\ &= r(\mathbf{v} \cdot \mathbf{z} + \mathbf{v} \cdot \mathbf{e}_j) \\ &= r(\mathbf{v} \cdot \mathbf{z} + v_j) \\ &= r^{(v_j)}(\mathbf{v} \cdot \mathbf{z}), \end{aligned}$$

it follows from (1) that

$$\frac{r_i}{r_i^{(v_j)}} = \frac{r_j}{r_j^{(v_i)}}$$

for all $i, j \in \{1, \dots, k\}$. Hence

$$\begin{aligned} \frac{t}{t^{(v_j)}} &= \prod_i^{w_1} \left(\frac{r_1}{r_1^{(v_j)}} \right)^{(v_1 i)} \prod_i^{w_2} \left(\frac{r_2}{r_2^{(v_j)}} \right)^{(v_1 w_1 + v_2 i)} \dots \\ &\quad \dots \prod_i^{w_k} \left(\frac{r_k}{r_k^{(v_j)}} \right)^{(v_1 w_1 + \dots + v_{k-1} w_{k-1} + v_k i)} \\ &= \prod_i^{w_1} \left(\frac{r_j}{r_j^{(v_1)}} \right)^{(v_1 i)} \prod_i^{w_2} \left(\frac{r_j}{r_j^{(v_2)}} \right)^{(v_1 w_1 + v_2 i)} \dots \\ &\quad \dots \prod_i^{w_k} \left(\frac{r_j}{r_j^{(v_k)}} \right)^{(v_1 w_1 + \dots + v_{k-1} w_{k-1} + v_k i)} \\ &= \frac{r_j^{(0)}}{r_j^{(v_1 w_1)}} \cdot \frac{r_j^{(v_1 w_1)}}{r_j^{(v_1 w_1 + v_2 w_2)}} \dots \frac{r_j^{(v_1 w_1 + \dots + v_{k-1} w_{k-1})}}{r_j^{(v_1 w_1 + \dots + v_k w_k)}} \\ &= \frac{r_j}{r_j^{(v_1 w_1 + \dots + v_k w_k)}} \\ &= \frac{r_j}{r_j^{(1)}} \end{aligned}$$

since $v_1 w_1 + \dots + v_k w_k = 1$.

We show that $r_j = c_j \prod_0^{v_j} t^{(i)}$. Since $r_j/r_j^{(1)} = t/t^{(v_j)}$, letting $T_j = \prod_0^{v_j} t^{(i)}$ it follows that $r_j/r_j^{(1)} = T_j/T_j^{(1)}$; hence $(r_j/T_j) = (r_j/T_j)^{(1)}$. It follows by Lemma 2.5.2 that $r_j/T_j = c_j$ for some $c_j \in K$. Thus, $R_j(z_1, \dots, z_k) = r_j(\mathbf{v} \cdot \mathbf{z}) = c_j \prod_0^{v_j} t^{(j)}(\mathbf{v} \cdot \mathbf{z}) = c_j \prod_0^{v_j} t(\mathbf{v} \cdot \mathbf{z} + j)$. \square

Lemma 2.8.2 summarizes the results of this section thus far.

Lemma 2.8.2 *Let $R_1, \dots, R_k \in K(z_1, \dots, z_k)$ be rational functions satisfying the relation*

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. For each subspace s of \mathbb{Q}^k let $R_{i,s} = \text{fix}_s R_i$.

- (1) If the dimension of s is less than $k - 1$, then there exists a rational function $r \in K(z_1, \dots, z_k)$ such that for all $i \in \{1, \dots, k\}$,

$$R_{i,s} = r/r^{e_i}.$$

- (2) If the dimension of s is $k - 1$ then there exists a univariate rational function $t \in K(z)$, $c_1, \dots, c_k \in K$, and a vector $\mathbf{v} \in \mathbb{Z}^k$ such that for all $i \in \{1, \dots, k\}$

$$R_{i,s} = c_i \prod_j^{v_i} t(\mathbf{v} \cdot \mathbf{z} + j).$$

- (3) If the dimension of s is k then $R_{i,s} \in K$ for all $i \in \{1, \dots, k\}$.
 (4) $R_i = \prod_s R_{i,s}$ for all $i \in \{1, \dots, k\}$, where the product is taken over all subspaces s of \mathbb{Q}^k .

Proof. Write

$$R_i = \frac{C}{C^{e_i}} \frac{D^{e_i}}{D} \frac{A_i}{B_i}$$

as in Lemma 2.1.3 and let $C_s = \text{fix}_s C$, $D_s = \text{fix}_s D$, $A_{i,s} = \text{fix}_s A_i$, and $B_{i,s} = \text{fix}_s B_i$. By Lemma 2.6.3

$$R_{i,s} = \frac{C_s}{C_s^{e_i}} \frac{D_s^{e_i}}{D_s} \frac{A_{i,s}}{B_{i,s}}.$$

(1) If the dimension of s is less than $k - 1$, then $A_{i,s} = B_{i,s} = 1$ since the irreducible divisors of A_i and B_i are simple. Hence $R_{i,s} = r/r^{e_i}$ where $r = C_s/D_s$.

(2) If the dimension of s is $k - 1$ then $R_{1,s}, \dots, R_{k,s}$ are simple by the corollary of Lemma 2.5.3. By Lemma 2.6.4

$$R_{i,s} R_{j,s}^{e_i} = R_{j,s} R_{i,s}^{e_j},$$

hence, by Lemma 2.8.1, there exist $c_1, \dots, c_k \in K$, $\mathbf{v} \in \mathbb{Z}^k$, and $t \in K(z)$ such that

$$R_{i,s} = c_i \prod_j^{v_i} t(\mathbf{v} \cdot \mathbf{z} + j)$$

for $i \in \{1, \dots, k\}$.

(3) By Lemma 2.5.2, $r_{i,s}$ is free of z_1, \dots, z_k , and thus constant.

(4) This is immediate from Lemma 2.5.8. \square

Lemma 2.8.3 *Let $R_1, \dots, R_k \in K(z_1, \dots, z_k)$ be rational functions satisfying the relation*

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all i and $j \in \{1, \dots, k\}$. There exist a rational function $r \in K(z_1, \dots, z_k)$, constants $c_1, \dots, c_k \in K$, finitely many vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{Z}^k$, and corresponding univariate rational functions $t_1, \dots, t_m \in K(z)$ such that for all $i \in \{1, \dots, k\}$,

$$R_i(\mathbf{z}) = c_i \frac{r(\mathbf{z})}{r^{e_i}(\mathbf{z})} \prod_{\ell=1}^m \prod_j^{v_{\ell,i}} t_{\ell}(\mathbf{v}_{\ell} \cdot \mathbf{z} + j).$$

Proof. For each subspace s of \mathbb{Q}^k and $i \in \{1, \dots, k\}$ let $R_{i,s} = \text{fix}_s R_i$. Let s_1, \dots, s_m be the subspaces s of \mathbb{Q}^k of dimension $k-1$ such that $R_{i,s} \neq 1$ for some $i \in \{1, \dots, k\}$. Let t_1, \dots, t_n be the subspaces of \mathbb{Q}^k of dimension less than $k-1$ such that $R_{i,s} \neq 1$ for some $i \in \{1, \dots, k\}$.

By Lemma 2.8.2 (2), for each ℓ , $1 \leq \ell \leq m$, $R_{i,s_{\ell}} = \prod_j^{v_{\ell,i}} t_{\ell}(\mathbf{v}_{\ell} \cdot \mathbf{z} + j)$ for some $\mathbf{v}_{\ell} \in \mathbb{Z}^k$ and $t_{\ell} \in K(z)$. By Lemma 2.8.2 (1), for each p , $1 \leq p \leq n$, there exists $r_p \in K(z_1, \dots, z_k)$ such that for $i \in \{1, \dots, k\}$, $R_{i,t_p} = r_p / r_p^{e_i}$. Let $r = \prod_{p=1}^n r_p$. Let $c_i = R_{i,\mathbb{Q}^k}$ for $i \in \{1, \dots, k\}$. By Lemma 2.8.2 (4),

$$\begin{aligned} R_{i,s} &= c_i R_{i,t_1} \cdots R_{i,t_n} R_{i,s_1} \cdots R_{i,s_m} \\ &= c_i \frac{r_{t_1}}{r_{t_1}^{e_i}} \cdots \frac{r_{t_n}}{r_{t_n}^{e_i}} \prod_{\ell=1}^m R_{i,s_{\ell}} \\ &= c_i \frac{r}{r^{e_i}} \prod_{\ell=1}^m \prod_j^{v_{\ell,i}} t_{\ell}(\mathbf{v}_{\ell} \cdot \mathbf{z} + j). \end{aligned}$$

\square

Theorem 2.8.4 [Ore–Sato] *Let $R_1, \dots, R_k \in K(\mathbf{z})$ be rational functions such that*

$$R_i R_j^{e_i} = R_j R_i^{e_j}$$

for all $i, j \in \{1, \dots, k\}$. Then there exist polynomials C and $D \in K[\mathbf{z}]$, a finite set $V \subset \mathbb{Z}^k$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ for each $\mathbf{v} \in V$ such that for all $i \in \{1, \dots, k\}$,

$$R_i(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{e}_i)}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{e}_i)} \prod_{\mathbf{v} \in V} \prod_j^{v_i} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}.$$

Proof. The c_i from Lemma 2.8.3 can be absorbed into the product by noting that

$$\prod_j^{e_i \cdot \mathbf{w}} c_i = \begin{cases} c_i & \text{if } \mathbf{w} = \mathbf{e}_i, \\ 1 & \text{if } \mathbf{w} \in \{\mathbf{e}_1, \dots, \mathbf{e}_k\} \setminus \{\mathbf{e}_i\}. \end{cases}$$

The rest is just a change of notation. \square

Corollary 2.8.5 For all $\mathbf{w} \in \mathbb{Z}^k$, let $R_{\mathbf{w}} \in K(\mathbf{z})$ be rational functions such that

$$R_{\mathbf{v}+\mathbf{w}} = R_{\mathbf{v}} R_{\mathbf{w}}^{\mathbf{v}}$$

for all \mathbf{w} and $\mathbf{v} \in \mathbb{Z}^k$. Then there exist polynomials C and $D \in K[\mathbf{z}]$, a finite set $V \subseteq \mathbb{Z}^k$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ for each $\mathbf{v} \in V$ such that for all $\mathbf{w} \in \mathbb{Z}^k$,

$$R_{\mathbf{w}}(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{w})}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{w})} \prod_{\mathbf{v} \in V} \prod_j^{v \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}.$$

Proof. By symmetry, $R_{\mathbf{v}+\mathbf{w}} = R_{\mathbf{w}} R_{\mathbf{v}}^{\mathbf{w}}$, hence, $R_{\mathbf{w}} R_{\mathbf{v}}^{\mathbf{w}} = R_{\mathbf{v}} R_{\mathbf{w}}^{\mathbf{v}}$. Letting $R_i = R_{\mathbf{e}_i}$ for $i \in \{1, \dots, k\}$, we have $R_i R_j^{e_i} = R_j R_i^{e_j}$. It follows by Theorem 2.8.4 that there exist polynomials $C, D \in K[\mathbf{z}]$, a finite set $V \in \mathbb{Z}$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}}$ for $\mathbf{v} \in V$ such that

$$R_{\mathbf{w}}(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{w})}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{w})} \prod_{\mathbf{v} \in V} \prod_j^{v \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}$$

for $\mathbf{w} \in \{\mathbf{e}_1, \dots, \mathbf{e}_k\}$. Let $\bar{R}_{\mathbf{w}}(\mathbf{z})$ be the right side of the last equation. Thus $R_{\mathbf{w}} = \bar{R}_{\mathbf{w}}$ for $\mathbf{w} \in \{\mathbf{e}_1, \dots, \mathbf{e}_k\}$.

We show that $R_{\mathbf{w}} = \bar{R}_{\mathbf{w}}$ for all $\mathbf{w} \in \mathbb{Z}^k$. It's clear that $R_{\mathbf{0}} = 1$ and $\bar{R}_{\mathbf{0}} = 1$ so $R_{\mathbf{0}} = \bar{R}_{\mathbf{0}}$. We show that $\bar{R}_{\mathbf{u}+\mathbf{w}} = \bar{R}_{\mathbf{u}} \bar{R}_{\mathbf{w}}^{\mathbf{u}}$.

$$\begin{aligned}
\bar{R}_u \bar{R}_w^u &= c^u c^w \frac{C^u}{C} \frac{D}{D^u} \left(\frac{C^w}{C} \frac{D}{D^w} \right)^u \prod_{v \in V} \prod_j^{v \cdot u} \frac{a_v(v \cdot z + j)}{b_v(v \cdot z + j)} \left(\prod_j^{v \cdot w} \frac{a_v(v \cdot z + j)}{b_v(v \cdot z + j)} \right)^u \\
&= c^{u+w} \frac{C^u}{C} \frac{D}{D^u} \frac{C^{w+u}}{C^u} \frac{D^u}{D^{w+u}} \prod_{v \in V} \prod_j^{v \cdot u} \frac{a_v(v \cdot z + j)}{b_v(v \cdot z + j)} \prod_j^{v \cdot w} \frac{a_v(v \cdot (z + u) + j)}{b_v(v \cdot (z + u) + j)} \\
&= c^{u+w} \frac{C^{u+w}}{C} \frac{D}{D^{u+w}} \prod_{v \in V} \prod_j^{v \cdot u} \frac{a_v(v \cdot z + j)}{b_v(v \cdot z + j)} \prod_j^{v \cdot w + v \cdot u} \frac{a_v(z \cdot v + j)}{b_v(z \cdot v + j)} \\
&= c^{u+w} \frac{C^{u+w}}{C} \frac{D}{D^{u+w}} \prod_{v \in V} \prod_j^{v \cdot (u+w)} \frac{a_v(z \cdot v + j)}{b_v(z \cdot v + j)} \\
&= \bar{R}_{u+w}.
\end{aligned}$$

Substituting $u - w$ for u in $R_{u+w} = R_u R_w^u$, it follows that $R_{u-w} = R_u / R_w^{u-w}$. Similarly $\bar{R}_{u-w} = \bar{R}_u / \bar{R}_w^{u-w}$. It follows that if $R_u = \bar{R}_u$ and $R_w = \bar{R}_w$ then $R_{u+w} = \bar{R}_{u+w}$ and $R_{u-w} = \bar{R}_{u-w}$. Thus, it follows by induction that $R_u = \bar{R}_u$ for all $u \in \mathbb{Z}^k$. \square

Chapter 3

The Multiplicative Structure of a Hypergeometric Term

The main results of this chapter are Theorem 3.7.1 and its corollary for algebraically closed fields, Corollary 3.7.3.

Theorem 3.7.1 Let f be an honest hypergeometric term on \mathbb{Z}^k . There exist relatively prime polynomials C and $D \in K[\mathbf{z}]$, a finite set $V \subset \mathbb{Z}^k$, univariate polynomials $a_{\mathbf{v}}, b_{\mathbf{v}} \in K[z]$, $\mathbf{v} \in V$ (all of which can be determined as in Theorem 3.1.8), and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_m$ such that

- (1) \mathbb{Z}^k is the disjoint union of the \mathcal{R}_i and a set of measure zero;
- (2) for each $i \in \{1, \dots, m\}$ there exists $\mathbf{z}_0 \in \mathcal{R}_i$ such that $C(\mathbf{z}_0) \neq 0$, and for all $\mathbf{z} \in \mathcal{R}_i$ for which $D(\mathbf{z}) \neq 0$,

$$f(\mathbf{z}) = f(\mathbf{z}_0) \frac{C(\mathbf{z})}{C(\mathbf{z}_0)} \frac{D(\mathbf{z}_0)}{D(\mathbf{z})} \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{z} \cdot \mathbf{v}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)}.$$

- (3) all the terms $a_{\mathbf{v}}(j)$ and $b_{\mathbf{v}}(j)$ occurring in the product are nonzero.

The hypergeometric terms that occur in practice can be expressed as products of Pochhammer symbols, so the question arises: Is this true in general? Corollary 3.7.3 show that if the field K is algebraically closed and the hypergeometric term is honest, then the answer is yes, at least piecewise.

Corollary 3.7.3 Let f be an honest hypergeometric term on \mathbb{Z}^k over a field K that is algebraically closed. Then there exist relatively prime polynomials C and $D \in K[\mathbf{z}]$ (as in Theorem 3.7.1) and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\boldsymbol{\gamma} \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that

(1) for all $\mathbf{z} \in \mathcal{R}_\ell$ such that $D(\mathbf{z}) \neq 0$,

$$f(\mathbf{z}) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} \frac{C(\mathbf{z})}{D(\mathbf{z})} \frac{\prod_{i=1}^p (m_i)_{\mathbf{v}_i \cdot \mathbf{z} + r_i}}{\prod_{j=1}^q (n_j)_{\mathbf{w}_j \cdot \mathbf{z} + s_j}};$$

(2) for all i and j and $\mathbf{z} \in \mathcal{R}_\ell$, $\mathbf{v}_i \cdot \mathbf{z} + r_i$ and $\mathbf{w}_j \cdot \mathbf{z} + s_j$ are positive;

(3) the Pochhammer symbols occurring in the products are nonzero.

3.1 Hypergeometric terms and term ratios

Definition 3.1.1 A *hypergeometric term* on \mathbb{Z}^k over a field K is a function $f: \mathbb{Z}^k \rightarrow K$ such that for $i \in \{1, \dots, k\}$ there exist nonzero polynomials A_1, \dots, A_k and $B_1, \dots, B_k \in K[\mathbf{z}]$ such that

$$A_i(\mathbf{z})f(\mathbf{z}) = B_i(\mathbf{z})f(\mathbf{z} + \mathbf{e}_i)$$

for every $\mathbf{z} \in \mathbb{Z}^k$.

Unfortunately, the definition of a hypergeometric term includes some pathological functions. For example, if f is any function $f: \mathbb{Z}^k \rightarrow K$ that is supported by the set of zeros of a nonzero polynomial p , then $p(\mathbf{z})f(\mathbf{z}) = 0$ for all \mathbf{z} , that is, $pf = 0$. It follows that $p^{\mathbf{e}_i} f^{\mathbf{e}_i} = 0$ for $i \in \{1, \dots, k\}$, and hence $pf = p^{\mathbf{e}_i} f^{\mathbf{e}_i}$ for $i \in \{1, \dots, k\}$. By Definition 3.1.1, $f(\mathbf{z})$ is a hypergeometric term. Such a hypergeometric term is called a *zero divisor*.

Definition 3.1.2 A function f on \mathbb{Z}^k is a *zero divisor* if there exists a nonzero polynomial $p \in K[\mathbf{z}]$ such that $pf = 0$.

Lemma 3.1.3 If f is a hypergeometric term on \mathbb{Z}^k , then for all $\mathbf{v} \in \mathbb{Z}^k$, there exist nonzero polynomials $A_{\mathbf{v}}$ and $B_{\mathbf{v}} \in K[\mathbf{z}]$ such

$$A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}.$$

Proof. We prove the lemma by induction. By the definition (3.1.1) of hypergeometric term, the statement of the lemma is true if $\mathbf{v} = \mathbf{e}_i$, $i \in \{1, \dots, k\}$,

and clearly the statement is true if $\mathbf{v} = \mathbf{0}$. We complete the induction by showing that if the statement is true for $\mathbf{v} = \mathbf{u}$ and $\mathbf{v} = \mathbf{w}$, then it is true for $\mathbf{v} = \mathbf{u} + \mathbf{w}$ and $\mathbf{v} = \mathbf{u} - \mathbf{w}$. Taking $A_{\mathbf{u}+\mathbf{w}} = A_{\mathbf{u}}A_{\mathbf{w}}^{\mathbf{u}}$ and $B_{\mathbf{u}+\mathbf{w}} = B_{\mathbf{u}}B_{\mathbf{w}}^{\mathbf{u}}$, we have $A_{\mathbf{u}+\mathbf{w}}f = A_{\mathbf{u}}A_{\mathbf{w}}^{\mathbf{u}}f = A_{\mathbf{w}}^{\mathbf{u}}B_{\mathbf{v}}f^{\mathbf{u}} = B_{\mathbf{u}}(A_{\mathbf{w}}f)^{\mathbf{u}} = B_{\mathbf{u}}(B_{\mathbf{w}}f^{\mathbf{w}})^{\mathbf{u}} = B_{\mathbf{u}}B_{\mathbf{w}}^{\mathbf{u}}f^{\mathbf{w}+\mathbf{u}} = B_{\mathbf{u}+\mathbf{w}}f^{\mathbf{u}+\mathbf{w}}$. Similarly, taking $A_{\mathbf{u}-\mathbf{w}} = A_{\mathbf{u}}B_{\mathbf{w}}^{\mathbf{u}-\mathbf{w}}$ and $B_{\mathbf{u}-\mathbf{w}} = B_{\mathbf{u}}A_{\mathbf{w}}^{\mathbf{u}-\mathbf{w}}$, $A_{\mathbf{u}-\mathbf{w}}f = A_{\mathbf{u}}B_{\mathbf{w}}^{\mathbf{u}-\mathbf{w}}f = B_{\mathbf{w}}^{\mathbf{u}-\mathbf{w}}B_{\mathbf{u}}f^{\mathbf{u}} = B_{\mathbf{u}}(B_{\mathbf{w}}f^{\mathbf{w}})^{\mathbf{u}-\mathbf{w}} = B_{\mathbf{u}}(A_{\mathbf{w}}f)^{\mathbf{u}-\mathbf{w}} = B_{\mathbf{u}}A_{\mathbf{w}}^{\mathbf{u}-\mathbf{w}}f^{\mathbf{u}-\mathbf{w}} = B_{\mathbf{u}-\mathbf{w}}f^{\mathbf{u}-\mathbf{w}}$.

□

Definition 3.1.4 For any hypergeometric term f on \mathbb{Z}^k and any $\mathbf{v} \in \mathbb{Z}^k$, a rational function $R_{\mathbf{v}} \in K(\mathbf{z})$ is a *term ratio in the direction \mathbf{v}* if

$$R_{\mathbf{v}} = \frac{A_{\mathbf{v}}}{B_{\mathbf{v}}}$$

for some nonzero polynomials $A_{\mathbf{v}}, B_{\mathbf{v}} \in K[\mathbf{z}]$ such that $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$.

Thus, by Lemma 3.1.3, a term ratio exists for each \mathbf{v} . Of course, we use the term *term ratio* because

$$\frac{f^{\mathbf{v}}(\mathbf{z})}{f(\mathbf{z})} = \frac{A_{\mathbf{v}}(\mathbf{z})}{B_{\mathbf{v}}(\mathbf{z})} = R_{\mathbf{v}}(\mathbf{z})$$

for all \mathbf{z} such that $f(\mathbf{z}) \neq 0$ and $B_{\mathbf{v}}(\mathbf{z}) \neq 0$. The following lemma shows that if f is not a zero divisor then the term ratio for f in the direction \mathbf{v} is unique.

Lemma 3.1.5 *If a hypergeometric term f on \mathbb{Z}^k is not a zero divisor, then for each $\mathbf{v} \in \mathbb{Z}^k$ there is a unique term ratio in the direction \mathbf{v} .*

Proof. Let $R_{\mathbf{v}}$ and $\bar{R}_{\mathbf{v}}$ be two term ratios for f in the direction \mathbf{v} . Then

$$R_{\mathbf{v}} = \frac{A_{\mathbf{v}}}{B_{\mathbf{v}}} \text{ and } \bar{R}_{\mathbf{v}} = \frac{\bar{A}_{\mathbf{v}}}{\bar{B}_{\mathbf{v}}}$$

where $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$ and $\bar{A}_{\mathbf{v}}f = \bar{B}_{\mathbf{v}}f^{\mathbf{v}}$. From the last two equations it follows that $(\bar{B}_{\mathbf{v}}A_{\mathbf{v}} - B_{\mathbf{v}}\bar{A}_{\mathbf{v}})f = 0$. Since f is not a zero divisor, this implies that $\bar{B}_{\mathbf{v}}A_{\mathbf{v}} - B_{\mathbf{v}}\bar{A}_{\mathbf{v}} = 0$, which implies that $R_{\mathbf{v}} = \bar{R}_{\mathbf{v}}$. □

Lemma 3.1.6 *Let f be a hypergeometric term on \mathbb{Z}^k that is not a zero divisor and let R_v be the term ratio for f in the direction v (which is unique by Lemma 3.1.5). Then*

$$R_{v+w} = R_v R_w^v = R_w R_v^w$$

for all $v, w \in \mathbb{Z}^k$.

Proof. For each $v \in \mathbb{Z}^k$, let A_v and B_v be polynomials such that $A_v f = B_v f^v$ as in Lemma 3.1.3. Then $A_v A_w^v f = A_w B_v f^v = B_v (A_w f)^v = B_v (B_w f^w)^v = B_v B_w^v f^{v+w}$. Thus, we have

$$A_{v+w} f = B_{v+w} f^{v+w}$$

and

$$A_v A_w^v f = B_v B_w^v f^{v+w}.$$

It follows from these equations that

$$(A_{v+w} B_v B_w^v - B_{v+w} A_v A_w^v) f = 0.$$

Since f is not a zero divisor, this implies that $A_{v+w} B_v B_w^v - B_{v+w} A_v A_w^v = 0$, from which it follows that $R_{v+w} = R_v R_w^v$. By symmetry, $R_{v+w} = R_w R_v^w$. \square

Proposition 3.1.7 *If a hypergeometric term f on \mathbb{Z}^k is a zero divisor, then for each $v \in \mathbb{Z}^k$ there exist nonzero A_v and $B_v \in K[z]$ such that $A_v = B_v f^v$ and $R_v = A_v / B_v$ satisfies $R_v R_w^v = R_w R_v^w$ for $v, w \in \mathbb{Z}^k$.*

Proof. Since $pf = 0$ for some polynomial $p \in K[z]$, $p^v f^v = 0$ for all $v \in \mathbb{Z}^k$. Thus, $pf = p^v f^v$ for all $v \in \mathbb{Z}^k$. Taking $A_v = p$ and $B_v = p^v$, it's easily verified that $R_v R_w^v = R_w R_v^w$ for all $v, w \in \mathbb{Z}^k$. \square

Theorem 3.1.8 *Let f be a hypergeometric term on \mathbb{Z}^k that is not a zero divisor. For all $w \in \mathbb{Z}^k$, let R_w be the term ratio of f in the direction w . Then there exist*

polynomials C and D in $K[\mathbf{z}]$, a finite set $V \subset \mathbb{Z}^k$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ for each $\mathbf{v} \in V$ such that

$$R_{\mathbf{w}}(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{w})}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{w})} \prod_{\mathbf{v} \in V} \prod_j^{v \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)} \text{ for all } \mathbf{w} \in \mathbb{Z}^k.$$

Proof. By Lemma 3.1.6, $R_{\mathbf{v}+\mathbf{w}} = R_{\mathbf{v}}R_{\mathbf{w}}^{\mathbf{v}}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^k$. The result follows immediately by Corollary 2.8.5. \square

3.2 Sets of measure zero

When considering sequences of one variable, it is useful to identify sequences that are equal except at a finite number of points. Thus, we identify sequences that are equal ‘almost everywhere’ and think of finite sets as ‘sets of measure zero’. When considering sequences of several variables, that is, functions from \mathbb{Z}^k to a field K , it is useful to consider the ‘sets of measure zero’ to be not finite sets, but sets that can be covered by a finite number of hyperplanes. Thus, in dimension 2, a ‘set of measure zero’ can be covered by a finite number of lines, and in three dimensions a ‘set of measure zero’ can be covered by a finite number of planes.

Definition 3.2.1 Define a *hyperplane* in \mathbb{Z}^k to be a set of the form $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} = n\}$, where $n \in \mathbb{Z}$ and \mathbf{v} is some vector in \mathbb{Z}^k . Define a *half-space* in \mathbb{Z}^k to be a set of the form $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} > n\}$, where $n \in \mathbb{Z}$ and \mathbf{v} is some vector in \mathbb{Z}^k . The boundary of the half-space $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} > n\}$ is the hyperplane $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} = n\}$. Since $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} < n\} = \{\mathbf{z}: -\mathbf{v} \cdot \mathbf{z} > -n\}$ is a half-space with boundary $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} = n\}$, each hyperplane is the boundary of two half-spaces, and \mathbb{Z}^k is the disjoint union of these two half-spaces and the boundary.

Definition 3.2.2 A *set of measure zero* in \mathbb{Z}^k is a set that can be covered by a finite number of hyperplanes. Two functions $f, g: \mathbb{Z}^k \rightarrow K$ are *equal almost everywhere* (written $f = g$ a.e.) if there exists a set S of measure zero such that $f(\mathbf{z}) = g(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{Z}^k \setminus S$. We say f is *degenerate* if $f = 0$ a.e. and f is *nondegenerate* if it is not degenerate.

Clearly, a finite union of sets of measure zero is a set of measure zero. Note that although we have defined a set of measure zero, we have not defined a measure. We use the phrase only because it is familiar and suggestive.

Lemma 3.2.3 *A nonzero simple polynomial $p \in K[\mathbf{z}]$ is nonzero except on a set of measure zero.*

Proof. Let $p(\mathbf{z}) = \bar{p}(\mathbf{z} \cdot \mathbf{v})$, where $\mathbf{v} \in \mathbb{Z}^k$ and $\bar{p} \in K[z]$ is a univariate polynomial. Let n_1, \dots, n_m be the integer roots of \bar{p} . If $p(\mathbf{z}_0) = 0$, then $\mathbf{z}_0 \cdot \mathbf{v}$ is a root of \bar{p} , and hence $\mathbf{z}_0 \cdot \mathbf{v} = n_i$ for some i . Hence \mathbf{z}_0 lies in one of the m hyperplanes $\{\mathbf{z}: \mathbf{z} \cdot \mathbf{v} = n_j\}$, $j = 1, \dots, m$, the union of which is a set of measure zero. \square

Lemma 3.2.4 *A function f is a hypergeometric term on \mathbb{Z}^k if and only if, for all $\mathbf{v} \in \mathbb{Z}^k$, there exist nonzero polynomials $A_{\mathbf{v}}$ and $B_{\mathbf{v}} \in K[\mathbf{z}]$ such that $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$ a.e.*

Proof. If f is a hypergeometric term, $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$ a.e. automatically. Conversely, if $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$ a.e., then $p_{\mathbf{v}}A_{\mathbf{v}}f = p_{\mathbf{v}}B_{\mathbf{v}}f^{\mathbf{v}}$, where $p_{\mathbf{v}} \in K[\mathbf{z}]$ is a product of nonzero linear polynomials. Hence f is a hypergeometric term. \square

Definition 3.2.5 *A hypergeometric term f on \mathbb{Z}^k is honest if for all $\mathbf{v} \in \mathbb{Z}^k$ there exist relatively prime polynomials $A_{\mathbf{v}}$ and $B_{\mathbf{v}} \in K[\mathbf{z}]$ such that $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$ a.e.*

3.3 Boxes

Definition 3.3.1 *A k -dimensional box of size n is a set of the form $\{\mathbf{z} \in \mathbb{Z}^k: c_i \leq z_i \leq c_i + n\}$, where $c_1, \dots, c_k \in \mathbb{Z}$. (A k -dimensional box is the set of integer points in a k -dimensional hypercube.)*

Lemma 3.3.2 *If a k -variable polynomial $p \in K[\mathbf{z}]$ is 0 on a k -dimensional box of size n , where n is the total degree of p , then p is identically 0.*

Proof. The lemma is clearly true if the degree of p is 0. We assume that it is true if the degree of p is $n - 1$, and show that it is true if the degree of p is

n . Suppose p is of degree n and is zero on a k -dimensional box of size n . For $i \in \{1, \dots, k\}$, $p - p^{e_i}$ is 0 on a k -dimensional box of size $n - 1$. By Lemma 2.0.2 $p - p^{e_i}$ is of degree at most $n - 1$, so $p - p^{e_i}$ is identically 0 by assumption. Thus, $p = p^{e_i}$ for all $i \in \{1, \dots, k\}$. By Lemma 2.5.2 p is free of z_i for $i \in \{1, \dots, k\}$. Thus p is constant, and hence p is identically zero. \square

Lemma 3.3.3 *If a hypergeometric term on \mathbb{Z}^k is not a zero divisor, then for any sequence $\mathbf{w}_1, \dots, \mathbf{w}_n$ of vectors in \mathbb{Z}^k the hypergeometric term $f^{\mathbf{w}_1} f^{\mathbf{w}_2} \dots f^{\mathbf{w}_n}$ is not a zero divisor.*

Proof. By Lemma 3.1.3, for any $\mathbf{v} \in \mathbb{Z}^k$ there exist nonzero polynomials $A_{\mathbf{v}}, B_{\mathbf{v}} \in K[\mathbf{z}]$ such that $A_{\mathbf{v}}f = B_{\mathbf{v}}f^{\mathbf{v}}$. If $f^{\mathbf{w}_1} f^{\mathbf{w}_2} \dots f^{\mathbf{w}_n}$ is a zero divisor, then for some nonzero polynomial p , $p f^{\mathbf{w}_1} f^{\mathbf{w}_2} \dots f^{\mathbf{w}_n} = 0$. Hence,

$$p B_{\mathbf{w}_1} \dots B_{\mathbf{w}_n} f^{\mathbf{w}_1} f^{\mathbf{w}_2} \dots f^{\mathbf{w}_n} = 0,$$

hence, $p A_{\mathbf{w}_1} \dots A_{\mathbf{w}_n} f^n = 0$, hence, $p A_{\mathbf{w}_1} \dots A_{\mathbf{w}_n} f = 0$, and hence, f is a zero divisor, contrary to assumption. \square

Lemma 3.3.4 *If f is a hypergeometric term on \mathbb{Z}^k , then either f is a zero divisor or f is nonzero on arbitrarily large k -dimensional boxes.*

Proof. Let V be the k -dimensional box $\{\mathbf{z} \in \mathbb{Z}^k : 0 \leq z_i \leq n\}$, and let $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. If f is not a zero divisor, by Lemma 3.3.3 $g = f^{\mathbf{v}_1} \dots f^{\mathbf{v}_m}$ is not a zero divisor, hence for some \mathbf{w} (infinitely many), $g(\mathbf{w}) \neq 0$, and hence $f(\mathbf{w} + \mathbf{v}_j) \neq 0$ for $j \in \{1, \dots, m\}$. Thus f is nonzero on the k -dimensional box

$$\{\mathbf{z} + \mathbf{w} : 0 \leq z_i \leq n, i = 1, \dots, k\},$$

which is of size n . \square

3.4 Polyhedral regions

Definition 3.4.1 A region $\mathcal{R} \subset \mathbb{Z}^k$ is *polyhedral* if $R = \mathbb{Z}^k$ or R is the intersection of a finite number of half-spaces of \mathbb{Z}^k .

Lemma 3.4.2 *The characteristic function of a half-space in \mathbb{Z}^k is a hypergeometric term; in fact, for all $w \in \mathbb{Z}^k$, $f = f^w$ a.e.*

Proof. Let the half-space be $\{z: v \cdot z > n\}$ for some $v \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$. Then $f(z) = f(z + w)$ unless $z \cdot v > n$ and $(z + w) \cdot v \leq n$, or unless $z \cdot v \leq n$ and $(z + w) \cdot v > n$; that is, $n < z \cdot v \leq n - v \cdot w$ or $n - v \cdot w < z \cdot v \leq n$. In either case, the set of exceptions is just the union of $|v \cdot w|$ hyperplanes. Thus $f(z) = f(z + w)$ a.e. \square

Lemma 3.4.3 *The characteristic function of a polyhedral region is a hypergeometric term.*

Proof. By definition, the region is the intersection of a finite number of half-spaces, so the characteristic function of the region is just the product of the characteristic functions of the half-spaces, which are all hypergeometric terms by Lemma 3.4.2. Since the product of two hypergeometric terms is a hypergeometric term, the characteristic function of the region is a hypergeometric term. \square

Lemma 3.4.4 *If \mathcal{R} is a polyhedral region in \mathbb{Z}^k , then either \mathcal{R} contains arbitrarily large k -dimensional boxes, or \mathcal{R} is a set of measure zero.*

Lemma 3.4.4 can be proved by elementary means, but we can't resist this short proof based on a result from Chapter 4.

Proof. Let f be the characteristic function of the region \mathcal{R} . By Lemma 3.4.3, f is a hypergeometric term. The function f is the product of characteristic functions of half-spaces, so f is holonomic. If f is a zero divisor, then by Lemma 4.1.2 $f = 0$ a.e., so \mathcal{R} is a set of measure 0. If f is not a zero divisor, then by Lemma 3.3.4, f is

nonzero on arbitrarily large k -dimensional boxes, and hence \mathcal{R} contains arbitrarily large k -dimensional boxes. \square

Definition 3.4.5 Let $\mathcal{R} \subset \mathbb{Z}^k$ be a polyhedral region. A *hypergeometric term* on \mathcal{R} over K is a function $f: \mathcal{R} \rightarrow K$ such that for $i \in \{1, \dots, k\}$ there exist nonzero polynomials $A_i, B_i \in K[z]$ such that

$$A_i(\mathbf{z})f(\mathbf{z}) = B_i(\mathbf{z})f(\mathbf{z} + \mathbf{e}_i)$$

for all \mathbf{z} such that \mathbf{z} and $\mathbf{z} + \mathbf{e}_i$ are in \mathcal{R} .

Lemma 3.4.6 If f is a hypergeometric term on a polyhedral region $\mathcal{R} \subset \mathbb{Z}^k$, then the function

$$g(\mathbf{z}) = \begin{cases} f(\mathbf{z}) & \mathbf{z} \in \mathcal{R} \\ 0 & \mathbf{z} \notin \mathcal{R} \end{cases}$$

is a hypergeometric term on \mathbb{Z}^k .

Proof. For $i \in \{1, \dots, k\}$ there exist nonzero polynomials A_i and B_i such that

$$A_i(\mathbf{z})f(\mathbf{z}) = B_i(\mathbf{z})f^{e_i}(\mathbf{z})$$

for all \mathbf{z} such that $\mathbf{z} \in \mathcal{R}$ and $\mathbf{z} + \mathbf{e}_i \in \mathcal{R}$. Let h be the characteristic function for \mathcal{R} . By Lemma 3.4.4, h is a hypergeometric term on \mathbb{Z}^k , so for $i \in \{1, \dots, k\}$ there exist nonzero polynomials \bar{A}_i and \bar{B}_i such that

$$\bar{A}_i(\mathbf{z})h(\mathbf{z}) = \bar{B}_i(\mathbf{z})h(\mathbf{z} + \mathbf{e}_i)$$

for all $\mathbf{z} \in \mathbb{Z}^k$. By considering separately the cases

- (1) $\mathbf{z} \in \mathcal{R}, \mathbf{z} + \mathbf{e}_i \in \mathcal{R}$,
- (2) $\mathbf{z} \in \mathcal{R}, \mathbf{z} + \mathbf{e}_i \notin \mathcal{R}$,
- (3) $\mathbf{z} \notin \mathcal{R}, \mathbf{z} + \mathbf{e}_i \in \mathcal{R}$, and
- (4) $\mathbf{z} \notin \mathcal{R}, \mathbf{z} + \mathbf{e}_i \notin \mathcal{R}$,

it is easily seen that

$$A_i \bar{A}_i g = B_i \bar{B}_i g^{e_i}$$

for $i \in \{1, \dots, k\}$. \square

3.5 Factorial hypergeometric terms

Definition 3.5.1 A hypergeometric term on \mathbb{Z}^k is *factorial* on a region $\mathcal{R} \subset \mathbb{Z}^k$ if there exist a finite set $V \subset \mathbb{Z}^k$, univariate polynomials a_v and $b_v \in K[z]$ and an integer n_v for each $v \in V$ such that, for all $z \in \mathcal{R}$,

- (1) $f(z) = \prod_{v \in V} \prod_{j=1}^{v \cdot z + n_v} \frac{a_v(j)}{b_v(j)}$,
- (2) for all $v \in V$, $v \cdot z + n_v$ is a positive integer, and
- (3) for all $v \in V$ and j , $1 \leq j \leq v \cdot z + n_v$, $a_v(j) \neq 0$ and $b_v(j) \neq 0$.

Recall that $\gamma^z = \gamma_1^{z_1} \cdots \gamma_k^{z_k}$ and $(m)_r = m(m+1) \cdots (m+r-1)$.

Lemma 3.5.2 If f is a hypergeometric term on \mathbb{Z}^k over a field that is algebraically closed and f is factorial on a region \mathcal{R} , then there exist a vector $\gamma \in K^k$, constants m_1, \dots, m_p and $n_1, \dots, n_q \in K$, vectors v_1, \dots, v_p and $w_1, \dots, w_q \in \mathbb{Z}^k$, and integers r_1, \dots, r_p and $s_1, \dots, s_q \in \mathbb{Z}$ such that, for all $z \in \mathcal{R}$,

- (1) $f(z) = \gamma^z \frac{\prod_{i=1}^p (m_i)_{v_i \cdot z + r_i}}{\prod_{j=1}^q (n_j)_{w_j \cdot z + s_j}}$;
- (2) $v_i \cdot z + r_i$ and $w_j \cdot z + s_j$ are positive integers for each $i \in \{1, \dots, p\}$ and $\{1, \dots, q\}$;
- (3) all the terms appearing in the numerator and denominator are nonzero.

Proof. Since f is factorial on \mathcal{R} ,

$$f(z) = \prod_{v \in V} \prod_{j=1}^{v \cdot z + n_v} \frac{a_v(j)}{b_v(j)}$$

for all $z \in \mathcal{R}$ where V , n_v , a_v , and b_v are as in Definition 3.5.1. Since a product of functions of the form (1) satisfying conditions (2) and (3) is of the same form (and likewise for reciprocals), we need only prove the lemma for

$$f(z) = \prod_{j=1}^{u \cdot z + \ell} a(j).$$

Since K is algebraically closed, $a(j) = \alpha(j - a_1) \cdots (j - a_t)$, where $\alpha, a_i \in K$. By

the same reasoning as before, we need only prove the lemma for

$$\prod_{j=1}^{\mathbf{u} \cdot \mathbf{z} + \ell} \alpha$$

and

$$\prod_{j=1}^{\mathbf{u} \cdot \mathbf{z} + \ell} (j - a_1).$$

The first is $\alpha^{\mathbf{u} \cdot \mathbf{z} + \ell} = \alpha^\ell \alpha^{u_1 z_1 + \dots + u_k z_k} = (\alpha^\ell)_1 \gamma^{\mathbf{z}}$, where $\gamma = (\alpha^{u_1}, \dots, \alpha^{u_k})$, and the second is $(1 - a_1)_{\mathbf{u} \cdot \mathbf{z} + \ell}$. We're given that $\mathbf{u} \cdot \mathbf{z} + \ell$ is positive on \mathcal{R} and $a(j)$ is nonzero for $1 \leq j \leq \mathbf{u} \cdot \mathbf{z} + \ell$, so conditions (2) and (3) follow. \square

Definition 3.5.3 A hypergeometric term f on \mathbb{Z}^k is *weakly factorial* on a region $\mathcal{R} \subset \mathbb{Z}^k$ if there exist a finite set $V \subset \mathbb{Z}^k$, univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ for each $\mathbf{v} \in V$, and $\mathbf{z}_0 \in \mathcal{R}$ such that, for all $\mathbf{z} \in \mathcal{R}$,

- (1) $f(\mathbf{z}) = \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{z} \cdot \mathbf{v}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)}$, and
- (2) for all $\mathbf{v} \in V$ and all j occurring in the product $\prod_j^{\mathbf{z} \cdot \mathbf{v}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)}$, $a_{\mathbf{v}}(j) \neq 0$ and $b_{\mathbf{v}}(j) \neq 0$.

An example of a hypergeometric term on \mathbb{Z} that is weakly factorial but not factorial is

$$f(z_1) = \prod_j^{z_1} (2j + 1) = \begin{cases} \prod_{j=0}^{z_1-1} (2j + 1), & z_1 \geq 0 \\ \prod_{j=1}^{-z_1} \frac{1}{1 - 2j}, & z_1 < 0. \end{cases}$$

Though f is not factorial, the domain of f can be split into two regions, $\{z_1 \geq 0\}$ and $\{z_1 < 0\}$, such that it is factorial on each region. The following lemma shows that this is true in general.

Lemma 3.5.4 *If a hypergeometric term is weakly factorial on a polyhedral region \mathcal{R} , then there exist a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_m$ such that $\mathcal{R} = \mathcal{R}_1 \cup \dots \cup \mathcal{R}_m$ and f is factorial on each of the \mathcal{R}_i .*

Proof. Since f is weakly factorial on \mathcal{R} ,

$$f(\mathbf{z}) = \prod_{\mathbf{v} \in V} \prod_{j \in \mathbf{v} \cdot \mathbf{z}_0}^{\mathbf{v} \cdot \mathbf{z}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)},$$

where $a_{\mathbf{v}}, b_{\mathbf{v}}, V$, and \mathbf{z}_0 are as in Definition 3.5.3. Let \mathcal{G} be the set of functions $g: V \rightarrow \{-1, 1\}$. We show that \mathcal{R} can be divided into at most $2^{|V|}$ regions $\{\mathcal{R}_g: g \in \mathcal{G}\}$, such that for each $\mathbf{v} \in V$, $g \in \mathcal{G}$, and region \mathcal{R}_g , there exist $\mathbf{w} = \pm \mathbf{v}$, $n = \pm \mathbf{v} \cdot \mathbf{z}_0$, univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}}$ which are nonzero for $1 \leq j \leq \mathbf{w} \cdot \mathbf{v} + n$, such that

$$(*) \quad \prod_{j \in \mathbf{v} \cdot \mathbf{z}_0}^{\mathbf{v} \cdot \mathbf{z}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)} = \prod_{j=1}^{\mathbf{w} \cdot \mathbf{z} + n} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)}.$$

We construct the region \mathcal{R}_g so that for every $\mathbf{v} \in V$, the sign of $\mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0)$ is constant over $\mathbf{z} \in \mathcal{R}_g$. For each $\mathbf{v} \in V$ define the half-spaces

$$\mathcal{H}_{\mathbf{v}}(1) = \{\mathbf{z}: \mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0) \geq 0\}$$

and

$$\mathcal{H}_{\mathbf{v}}(-1) = \{\mathbf{z}: \mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0) < 0\}.$$

For each $g \in \mathcal{G}$, the region \mathcal{R}_g is defined by

$$\mathcal{R}_g = \mathcal{R} \cap \bigcap_{\mathbf{v} \in V} \mathcal{H}_{\mathbf{v}}(g(\mathbf{v})).$$

Clearly $\mathcal{R} = \bigcup_g \mathcal{R}_g$ and \mathcal{R}_g is polyhedral. Note $|\mathcal{G}| = 2^{|V|}$, so $|\{\mathcal{R}_g: g \in \mathcal{G}\}| \leq 2^{|V|}$. By definition, the sign of $\mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0)$ is constant on $\mathcal{H}_{\mathbf{v}}(g(\mathbf{v}))$, so, for each \mathbf{v} , the sign of $\mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0)$ is constant for $\mathbf{z} \in \mathcal{R}_g$.

We need to show that for a given region (*) is true. Let a region \mathcal{R}_g and a vector \mathbf{v} be given. If $\mathbf{v} \cdot \mathbf{z} - \mathbf{v} \cdot \mathbf{z}_0 = \mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0)$ is nonnegative on \mathcal{R}_g , then

$$\prod_{j \in \mathbf{v} \cdot \mathbf{z}_0}^{\mathbf{v} \cdot \mathbf{z}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)} = \prod_{j=\mathbf{v} \cdot \mathbf{z}_0}^{\mathbf{v} \cdot \mathbf{z}-1} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)} = \prod_{j=1}^{\mathbf{v} \cdot \mathbf{z}-\mathbf{v} \cdot \mathbf{z}_0} \frac{a_{\mathbf{v}}(j + \mathbf{v} \cdot \mathbf{z}_0 - 1)}{b_{\mathbf{v}}(j + \mathbf{v} \cdot \mathbf{z}_0 - 1)} = \prod_{j=1}^{\mathbf{w} \cdot \mathbf{z} + n_{\mathbf{v}}} \frac{\bar{a}_{\mathbf{v}}(j)}{\bar{b}_{\mathbf{v}}(j)},$$

where $\bar{a}_{\mathbf{v}}(j) = a_{\mathbf{v}}(j + \mathbf{v} \cdot \mathbf{z}_0 - 1)$, $\bar{b}_{\mathbf{v}}(j) = b_{\mathbf{v}}(j + \mathbf{v} \cdot \mathbf{z}_0 - 1)$, $\mathbf{w} = \mathbf{v}$, and $n_{\mathbf{v}} = -\mathbf{v} \cdot \mathbf{z}_0$.

If $\mathbf{v} \cdot \mathbf{z} - \mathbf{v} \cdot \mathbf{z}_0 = \mathbf{v} \cdot (\mathbf{z} - \mathbf{z}_0)$ is negative on \mathcal{R}_g ,

$$\begin{aligned} \prod_j^{\mathbf{v} \cdot \mathbf{z}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)} &= \prod_{j=\mathbf{v} \cdot \mathbf{z}_0}^{\mathbf{v} \cdot \mathbf{z} - 1} \frac{b_{\mathbf{v}}(j)}{a_{\mathbf{v}}(j)} \\ &= \prod_{j=1-\mathbf{v} \cdot \mathbf{z}_0}^{-\mathbf{v} \cdot \mathbf{z}} \frac{b_{\mathbf{v}}(-j)}{a_{\mathbf{v}}(-j)} \\ &= \prod_{j=1}^{\mathbf{v} \cdot \mathbf{z}_0 - \mathbf{v} \cdot \mathbf{z}} \frac{b_{\mathbf{v}}(-j - \mathbf{v} \cdot \mathbf{z}_0)}{a_{\mathbf{v}}(-j - \mathbf{v} \cdot \mathbf{z}_0)} \\ &= \prod_{j=1}^{\mathbf{w} \cdot \mathbf{z} + n_{\mathbf{v}}} \frac{\bar{a}_{\mathbf{v}}(j)}{\bar{b}_{\mathbf{v}}(j)}, \end{aligned}$$

where $\bar{a}_{\mathbf{v}}(j) = b_{\mathbf{v}}(-j - \mathbf{v} \cdot \mathbf{z}_0)$ and $\bar{b}_{\mathbf{v}}(j) = a_{\mathbf{v}}(-j - \mathbf{v} \cdot \mathbf{z}_0)$, $\mathbf{w} = -\mathbf{v}$, and $n_{\mathbf{v}} = \mathbf{v} \cdot \mathbf{z}_0$. □

3.6 Path connected regions

Recall that a lattice path in \mathbb{Z}^k is a sequence $\{T_i\}_{i \geq 1}$ in \mathbb{Z}^k such that $T_i - T_{i-1} \in \{\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_k\}$ for all $i > 1$.

Definition 3.6.1 Let \mathcal{R} and $\bar{\mathcal{R}} \in \mathbb{Z}^k$. The region \mathcal{R} is *lattice path connected in* $\bar{\mathcal{R}}$ if for all \mathbf{z}_1 and $\mathbf{z}_2 \in \mathcal{R}$ there exists a lattice path $\{T_i\}_{i \geq 1}$ contained in $\bar{\mathcal{R}}$ such that $T_1 = \mathbf{z}_1$ and $T_i = \mathbf{z}_2$ for some positive integer i . The region \mathcal{R} is *lattice path connected* if it is lattice path connected in itself.

Definition 3.6.2 For any subsets S , \mathcal{R} , and $\bar{\mathcal{R}} \subset \mathbb{Z}^k$, the region \mathcal{R} is *S-path connected in* $\bar{\mathcal{R}}$ if for all \mathbf{z}_1 and $\mathbf{z}_2 \in \mathcal{R}$ there exists a sequence $\{T_i\}_{i \geq 1}$ contained in $\bar{\mathcal{R}}$ such that $T_1 = \mathbf{z}_1$ and $T_i = \mathbf{z}_2$ for some positive integer i , and $T_i - T_{i-1} \in S$ for all $i > 1$.

Thus a region is lattice path connected if it is S -path connected in itself for $S = \{\pm \mathbf{e}_1, \dots, \mathbf{e}_k\}$.

Lemma 3.6.3 Let $\mathcal{R} \subset \mathbb{Z}^k$ be a polyhedral region. For any positive integer n , \mathcal{R} can be written as the union of a set of measure zero and a polyhedral region \mathcal{R}'

such that every $\mathbf{z} \in \mathcal{R}'$ lies in a k -dimensional box of size n contained entirely in \mathcal{R} .

Proof. Let $\mathcal{R}' = \cap_{\mathbf{v} \in B} (R - \mathbf{v})$ where $B = \{\mathbf{b} \in \mathbb{Z}^k : \text{for each } i \in \{1, \dots, k\}, 0 \leq b_i \leq n\}$. Clearly \mathcal{R}' is a polyhedral region and for every $\mathbf{z} \in \mathcal{R}'$ the box $\mathbf{z} + B = \{\mathbf{z} + \mathbf{b} : \mathbf{b} \in B\}$ is contained in \mathcal{R} , so we need to show that $S = \mathcal{R} \setminus \mathcal{R}'$ is a set of measure zero.

Since \mathcal{R} is the intersection of a finite number of half-spaces, there exists a finite set $T \subset \mathbb{Z}^k \times \mathbb{Z}$ such that $\mathbf{z} \in \mathcal{R}$ if and only if $\mathbf{z} \in \mathbb{Z}^k$ and $\mathbf{v} \cdot \mathbf{z} \leq n$ for all $(\mathbf{v}, n) \in T$. Thus, $\mathbf{z} \in S$ implies there exists $(\mathbf{v}, n) \in T$ and $\mathbf{b} \in B$ such that $\mathbf{v} \cdot \mathbf{z} \leq n$ and $\mathbf{v} \cdot (\mathbf{z} + \mathbf{b}) > n$, which in turn implies $\mathbf{v} \cdot \mathbf{z} = m$ for some integer m such that $n - \mathbf{v} \cdot \mathbf{b} < m \leq n$. It follows that S can be covered by hyperplanes of the form $\{\mathbf{z} : \mathbf{v} \cdot \mathbf{z} = m\}$, where $\mathbf{b} \in B$, $(\mathbf{v}, n) \in T$ and $n - \mathbf{v} \cdot \mathbf{b} < m \leq n$. Clearly, there are a finite number of such hyperplanes, so S is a set of measure zero. \square

Lemma 3.6.4 *Let B_0 and B_1 be k -dimensional boxes of size 1, let \bar{S} be the convex hull of B_0 and B_1 in \mathbb{Q}^k , and let S be the set of integer points in \bar{S} . Then S is lattice path connected.*

Proof. Let M be a $k \times k$ matrix with entries equal to ± 1 on the diagonal and 0 off the diagonal. It is easily seen that the lemma is true for the boxes B_0 and B_1 if and only if it is true for the reflections $MB_0 = \{Mb : b \in B_0\}$ and $MB_1 = \{Mb : b \in B_1\}$ of B_0 and B_1 . Similarly, letting $\mathbf{v} \in \mathbb{Z}^k$, the lemma is true for B_0 and B_1 if and only if it is true for the translations $B_0 + \mathbf{v} = \{\mathbf{b} + \mathbf{v} : \mathbf{b} \in B_0\}$ and $B_1 + \mathbf{v} = \{\mathbf{b} + \mathbf{v} : \mathbf{b} \in B_1\}$ of B_0 and B_1 . By applying a sequence of such transformations, we may assume that B_0 is the unit k -dimensional box at the origin in the first orthant, $B_0 = \{\mathbf{z} \in \mathbb{Z}^k : \text{for each } i \in \{1, \dots, k\}, 0 \leq z_i \leq 1\}$, and that $B_1 = B_0 + \mathbf{w}$ where $\mathbf{w} \in \mathbb{Z}^k$ and $w_i \geq 0$ for each $i \in \{1, \dots, k\}$.

Further, if $w_i = 0$ for some $i \in \{1, \dots, k\}$, then the lemma follows easily by induction from the case of dimension $k - 1$. Since the case of dimension 1 is trivial,

we may assume that $w_i > 0$ for each $i \in \{1, \dots, k\}$.

Let \bar{B}_0 be the convex hull of B_0 in \mathbb{Q}^k . Thus, $\bar{B}_0 = \{z \in \mathbb{Q}^k : \text{for each } i \in \{1, \dots, k\}, 0 \leq z_i \leq 1\}$.

We claim that any $z \in S$ can be written in the form $z = z' + sw$, where $z' \in \bar{B}_0$, $s \in \mathbb{Q}$, and $s \geq 0$. Since $z \in S$,

$$z = \sum_{b \in B_0} \lambda_b b + \sum_{b \in B_1} \lambda_b b \quad ,$$

where

$$\sum_{b \in B_0} \lambda_b + \sum_{b \in B_1} \lambda_b = 1 \quad .$$

Hence,

$$\begin{aligned} z &= \sum_{b \in B_0} \lambda_b b + \sum_{b \in B_0} \lambda_{b+w} (b + w) \\ &= \sum_{b \in B_0} (\lambda_b + \lambda_{b+w}) b + \sum_{b \in B_0} \lambda_{b+w} w \\ &= z' + sw \quad , \end{aligned}$$

where

$$z' = \sum_{b \in B_0} (\lambda_b + \lambda_{b+w}) b$$

and

$$s = \sum_{b \in B_0} \lambda_{b+w} \geq 0 \quad .$$

Since

$$\sum_{b \in B_0} (\lambda_b + \lambda_{b+w}) = \sum_{b \in B_0} \lambda_b + \sum_{b \in B_1} \lambda_b = 1 \quad ,$$

it follows that $z' \in \bar{B}_0$.

We claim that for any $z \in S \setminus B_1$, there exists $j \in \{1, \dots, k\}$ such that $z + e_j \in S$. Let $z = z' + sw$ where $z' \in \bar{B}_0$, $s \in \mathbb{Q}$ and $s \geq 0$. For each $i \in \{1, \dots, k\}$, the line $\{z' - rw : r \in \mathbb{Q}\}$ intersects the hyperplane $z \cdot e_i = 0$ at a point $v_i = z' - r_i w$ where $r_i \in \mathbb{Q}$ and $r_i \geq 0$. Thus, $v_{i,i} = 0$ for each $i \in \{1, \dots, k\}$.

Let $j \in \{1, \dots, k\}$ be such that r_j is minimal. By the minimality of r_j and the positivity of w_i , it follows that $v_{j,i} \geq v_{i,i} = 0$ for $i \neq j$. Thus, $v_{j,i} \geq 0$ for all $i \in \{1, \dots, k\}$. Further, since $0 \leq z'_i \leq 1$, and $r_i \geq 0$, it follows that $0 \leq v_{j,i} \leq 1$ for all $i \in \{1, \dots, k\}$ and $v_{j,j} = 0$. Letting $t = r + s$, it follows that $\mathbf{z} = \mathbf{v}_j + t\mathbf{w}$, and, hence,

$$\mathbf{z} + \mathbf{e}_j = (1 - t)(\mathbf{v}_j + \mathbf{e}_j) + t(\mathbf{v}_j + \mathbf{e}_j + \mathbf{w}) \quad .$$

It is easily seen that $t < 1$, otherwise $\mathbf{z} \in B_1$. Since $v_{j,j} = 0$ and $\mathbf{v}_j \in \bar{B}_0$, it follows that $\mathbf{v}_j + \mathbf{e}_j \in \bar{B}_0$, and, hence, $\mathbf{v}_j + \mathbf{e}_j + \mathbf{w} \in \bar{B}_1$. Thus, $\mathbf{z} + \mathbf{e}_j \in \bar{S}$ as claimed.

We show that every $\mathbf{z} \in S$ is lattice path connected to a point in B_1 . Suppose to the contrary some $\mathbf{z} \in S \setminus B_1$ is not lattice path connected to a point in B_1 . Assume that $\sum_{i=1}^k z_i$ is maximal. Such a \mathbf{z} exists since $\sum_{i=1}^k z_i < \sum_{i=1}^k w_i$ for all $\mathbf{z} \in S \setminus B_1$. For some $j \in \{1, \dots, k\}$, we have $\mathbf{z} + \mathbf{e}_j \in S$. By the maximality of $\sum_{i=1}^k z_i$, it follows that $\mathbf{z} + \mathbf{e}_j \in B_1$. Thus, \mathbf{z} is lattice path connected to B_1 contrary to assumption. Clearly B_1 is lattice path connected, so S is lattice path connected.

□

Corollary 3.6.5 *Let $\mathcal{R} \subset \mathbb{Z}^k$ be a polyhedral region and let \mathbf{z}_1 and $\mathbf{z}_2 \in R$. If \mathbf{z}_1 and \mathbf{z}_2 are contained in k -dimensional boxes of size 1 contained entirely in R , then there exists a lattice path from \mathbf{z}_1 to \mathbf{z}_2 contained entirely in R .*

Proof. Let B_1 and $B_2 \subset \mathcal{R}$ be k -dimensional boxes containing \mathbf{z}_1 and \mathbf{z}_2 respectively. Let \bar{S} be the convex hull of B_0 and B_1 in \mathbb{Q}^k and let S be the set of integer points in \bar{S} . By the definition of a polyhedral region there exists a convex subset \bar{R} of \mathbb{Q}^k such that R is the set of integer points in \bar{R} . Since \bar{R} is convex, it contains the convex hull of any of its subsets. Thus, $\bar{S} \subset \bar{R}$ and, hence, $S \subset R$. But S is lattice path connected by Lemma 3.6.4. □

3.7 Structure theorem for honest hypergeometric terms

Theorem 3.7.1 *Let f be an honest hypergeometric term on \mathbb{Z}^k . There exist relatively prime polynomials C and $D \in K[z]$, a finite set $V \subset \mathbb{Z}^k$, univariate polynomials $a_v, b_v \in K[z]$, $v \in V$ (all of which can be determined as in Theorem 3.1.8), and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_m$ such that*

- (1) \mathbb{Z}^k is the disjoint union of the \mathcal{R}_i and a set of measure zero;
- (2) for each $i \in \{1, \dots, m\}$ there exists $z_0 \in \mathcal{R}_i$ such that $C(z_0) \neq 0$, and for all $z \in \mathcal{R}_i$ for which $D(z) \neq 0$,

$$f(z) = f(z_0) \frac{C(z)}{C(z_0)} \frac{D(z_0)}{D(z)} \prod_{v \in V} \prod_j^{z \cdot v} \frac{a_v(j)}{b_v(j)}.$$

- (3) all the terms $a_v(j)$ and $b_v(j)$ occurring in the product are nonzero.

We conjecture that Theorem 3.7.1 is true for hypergeometric terms that are not honest if the condition that C and D are relatively prime is dropped.

Theorem 3.7.1 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

Proof. If f is a zero divisor and $pf = 0$, then the lemma is true with $D = p$, $C = 1$, V empty, and one region $\mathcal{R}_1 = \mathbb{Z}^k$. We assume hence forth that f is not a zero divisor. Let R_w be the term ratio of f in the direction w . By Theorem 3.1.8,

$$R_w(z) = \frac{C(z+w)}{C(z)} \frac{D(z)}{D(z+w)} \prod_{v \in V} \prod_0^{v \cdot w} \frac{a_v(z \cdot v + j)}{b_v(z \cdot v + j)}$$

for all $w \in \mathbb{Z}^k$. We may assume C and D are relatively prime. Let \bar{A}_w and $\bar{B}_w \in K[z]$ be relatively prime polynomials such that

$$(1) \quad \frac{\bar{A}_w}{\bar{B}_w} = \prod_{v \in V} \prod_0^{v \cdot w} \frac{a_w(z \cdot v + j)}{b_w(z \cdot v + j)}.$$

Thus \bar{A}_w and \bar{B}_w are products of simple polynomials, and by Lemma 3.2.3 \bar{A}_w and \bar{B}_w are nonzero except on a set of measure zero.

Since f is honest, for each $\mathbf{w} \in \mathbb{Z}$ there exist relatively prime polynomials $A_{\mathbf{w}}$ and $B_{\mathbf{w}} \in K[\mathbf{z}]$ such that $A_{\mathbf{w}}f = B_{\mathbf{w}}f^{\mathbf{w}}$ a.e. Since f is not a zero divisor, by Lemma 3.1.5 the term ratio $R_{\mathbf{w}}$ is unique. Hence,

$$\frac{A_{\mathbf{w}}}{B_{\mathbf{w}}} = R_{\mathbf{w}} = \frac{C^{\mathbf{w}}D\bar{A}_{\mathbf{w}}}{CD^{\mathbf{w}}\bar{B}_{\mathbf{w}}}$$

for all $\mathbf{w} \in \mathbb{Z}^k$ and, hence, there exist relatively prime polynomials $p_{\mathbf{w}}$ and $q_{\mathbf{w}} \in K[\mathbf{z}]$ such that

$$p_{\mathbf{w}}A_{\mathbf{w}} = q_{\mathbf{w}}C^{\mathbf{w}}D\bar{A}_{\mathbf{w}} \text{ and } p_{\mathbf{w}}B_{\mathbf{w}} = q_{\mathbf{w}}CD^{\mathbf{w}}\bar{B}_{\mathbf{w}} \quad .$$

Since $p_{\mathbf{w}}$ and $q_{\mathbf{w}}$ are relatively prime, it follows that $q_{\mathbf{w}} \mid A_{\mathbf{w}}$ and $q_{\mathbf{w}} \mid B_{\mathbf{w}}$, and since $A_{\mathbf{w}}$ and $B_{\mathbf{w}}$ are relatively prime, it follows that $q_{\mathbf{w}}$ must be trivial. Hence, for each $\mathbf{w} \in \mathbb{Z}^k$,

$$C^{\mathbf{w}}D\bar{A}_{\mathbf{w}}f = CD^{\mathbf{w}}\bar{B}_{\mathbf{w}}f^{\mathbf{w}} \text{ a.e.}$$

Let d be the total degree of CD . By Lemma 3.3.2, CD is nonzero for at least one point in any k -dimensional box of size d . Let $S_0 \subset \mathbb{Z}^k$ be the union of all k -dimensional boxes of size d containing 0. Let $S_1 = \{\mathbf{s} \pm \mathbf{e}_i : \mathbf{s} \in S_0 \text{ and } i \in \{1, \dots, k\}\}$. Thus, S_1 is the union of all k -dimensional boxes of size d containing a point $\pm \mathbf{e}_i$, $i \in \{1, \dots, k\}$. Finally, let $S = S_0 - S_1 = \{\mathbf{s}_0 - \mathbf{s}_1 : \mathbf{s}_0 \in S_0, \mathbf{s}_1 \in S_1\}$. For each $\mathbf{w} \in \mathbb{Z}^k$, the polynomials $\bar{A}_{\mathbf{w}}$ and $\bar{B}_{\mathbf{w}}$ are nonzero except on a set of measure zero, and $\bar{A}_{\mathbf{w}}C^{\mathbf{w}}Df = \bar{B}_{\mathbf{w}}CD^{\mathbf{w}}f^{\mathbf{w}}$ except on a set of measure zero. Call the union of these two sets of measure zero $H_1(\mathbf{w})$ and let $H_1 = \cup_{\mathbf{w} \in S} H_1(\mathbf{w})$. Since S is finite, it follows that H_1 is a set of measure zero. Thus, for all $\mathbf{w} \in S$ and all $\mathbf{z} \notin H_1$,

$$(2) \quad \bar{A}_{\mathbf{w}}(\mathbf{z}) \neq 0, \bar{B}_{\mathbf{w}}(\mathbf{z}) \neq 0, \text{ and}$$

$$(3) \quad \bar{A}_{\mathbf{w}}(\mathbf{z})C(\mathbf{z} + \mathbf{w})D(\mathbf{z})f(\mathbf{z}) = \bar{B}_{\mathbf{w}}(\mathbf{z})C(\mathbf{z})D(\mathbf{z} + \mathbf{w})f(\mathbf{z} + \mathbf{w}).$$

Let H_2 be a finite set of hyperplanes covering H_1 . The hyperplanes in H_2 divide space into a finite number of polyhedral regions $\bar{\bar{\mathcal{R}}}_i$ such that \mathbb{Z}^k is the

disjoint union of the regions and the union of the hyperplanes. (The hyperplanes are not necessarily disjoint, but the regions are disjoint from each other and from the hyperplanes.) Further, by Lemma 3.6.3 each region $\bar{\bar{\mathcal{R}}}_i$ can be written as a union of a polyhedral region $\bar{\mathcal{R}}_i$ and a set of measure zero such that each $\mathbf{z} \in \bar{\mathcal{R}}_i$ is contained in a k -dimensional box of size d contained entirely in $\bar{\bar{\mathcal{R}}}_i$. By Lemma 3.6.4 each $\bar{\mathcal{R}}_i$ can be written as a union of a polyhedral region \mathcal{R}_i and a set of measure zero such that \mathcal{R}_i is lattice path connected in $\bar{\mathcal{R}}_i$. Thus,

- (4) $\mathbb{Z}^k = \mathcal{R}_1 \cup \dots \cup \mathcal{R}_m \cup H$, where H is a set of measure zero;
- (5) $\bar{A}_{\mathbf{w}} C^{\mathbf{w}} D f = \bar{B}_{\mathbf{w}} C D^{\mathbf{w}} f^{\mathbf{w}}$ for every $\mathbf{z} \in \bar{\bar{\mathcal{R}}}_i$, $i \in \{1, \dots, m\}$, and every $\mathbf{w} \in S$;
- (6) $\bar{A}_{\mathbf{w}}$ and $\bar{B}_{\mathbf{w}}$ are nonzero on $\bar{\bar{\mathcal{R}}}_i$, $i \in \{1, \dots, m\}$, for every $\mathbf{w} \in S$;
- (7) \mathcal{R}_i is lattice path connected in $\bar{\mathcal{R}}_i$ for $i \in \{1, \dots, m\}$;
- (8) for every $\mathbf{z} \in \bar{\mathcal{R}}_i$, \mathbf{z} is contained in a k -dimensional box of size d contained in $\bar{\bar{\mathcal{R}}}_i$, $i \in \{1, \dots, m\}$.

From (1), (5) and (6) it follows that

$$(9) \quad f(\mathbf{z}_2) C(\mathbf{z}_1) D(\mathbf{z}_2) = f(\mathbf{z}_1) C(\mathbf{z}_2) D(\mathbf{z}_1) \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{z}_2 \cdot \mathbf{z}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)} \text{ if } \mathbf{z}_1 \in \bar{\bar{\mathcal{R}}}_i \text{ and } \mathbf{z}_2 - \mathbf{z}_1 \in S, i \in \{1, \dots, m\}.$$

Since any \mathcal{R}_i of measure zero can be absorbed into H we may assume \mathcal{R}_i is not a set of measure zero for $i \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$ let $\mathcal{R}'_i = \{\mathbf{z} \in \mathcal{R}_i : C(\mathbf{z}) D(\mathbf{z}) \neq 0\}$, and let $\bar{\mathcal{R}}'_i = \{\mathbf{z} \in \bar{\mathcal{R}}_i : C(\mathbf{z}) D(\mathbf{z}) \neq 0\}$. Since \mathcal{R}_i is not a set of measure zero, \mathcal{R}_i contains a k -dimensional box of size d by Lemma 3.4.4. Hence, by Lemma 3.3.2, $C(\mathbf{z}) D(\mathbf{z})$ is nonzero for some $\mathbf{z} \in \mathcal{R}_i$, and hence \mathcal{R}'_i is not empty.

We claim \mathcal{R}'_i is S -path connected in $\bar{\mathcal{R}}'_i$ for each $i \in \{1, \dots, m\}$. Let \mathbf{z}_1 and $\mathbf{z}_2 \in \mathcal{R}'_i$ be given. We construct a sequence $\mathbf{v}'_1, \dots, \mathbf{v}'_{\ell}$ in $\bar{\mathcal{R}}'_i$ such that $\mathbf{v}'_1 = \mathbf{z}_1$, $\mathbf{v}'_{\ell} = \mathbf{z}_2$, and $\mathbf{v}'_j - \mathbf{v}'_{j-1} \in S$ for $j = 2, \dots, \ell$. Since \mathcal{R}_i is lattice path connected in $\bar{\mathcal{R}}_i$, there exists a sequence $\mathbf{v}_1, \dots, \mathbf{v}_{\ell}$ in $\bar{\mathcal{R}}_i$ such that $\mathbf{v}_1 = \mathbf{z}_1$, $\mathbf{v}_{\ell} = \mathbf{z}_2$, and $\mathbf{v}_j - \mathbf{v}_{j-1} \in \{\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_k\}$. For each $j \in \{1, \dots, \ell\}$ there is a k -dimensional

box of size d containing \mathbf{v}_j contained in $\bar{\bar{\mathcal{R}}}_i$. This box contains a point \mathbf{v}'_j such that $C(\mathbf{v}'_j)D(\mathbf{v}'_j) \neq 0$, since the total degree of CD is d . Obviously, we can take $\mathbf{v}'_1 = \mathbf{z}_1$ and $\mathbf{v}'_\ell = \mathbf{z}_2$. By the construction of S , it's clear that $\mathbf{v}'_j - \mathbf{v}'_{j-1} \in S$ for $j \in \{2, \dots, \ell\}$. Thus, \mathcal{R}'_i is S -path connected in $\bar{\bar{\mathcal{R}}}'_i$ as claimed.

For all \mathbf{z}_1 and $\mathbf{z}_2 \in \bar{\bar{\mathcal{R}}}'_i$, define

$$g(\mathbf{z}_1, \mathbf{z}_2) = \frac{C(\mathbf{z}_2)}{C(\mathbf{z}_1)} \frac{D(\mathbf{z}_1)}{D(\mathbf{z}_2)} \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{z}_2 \cdot \mathbf{v}}_{\mathbf{z}_1 \cdot \mathbf{v}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)}.$$

We claim that for all \mathbf{z}_1 and $\mathbf{z}_2 \in \mathcal{R}'_i$,

$$f(\mathbf{z}_2) = f(\mathbf{z}_1)g(\mathbf{z}_1, \mathbf{z}_2).$$

We show by induction on j that if $\{\mathbf{z}_\nu\}_{\nu \geq 1}$ is a sequence in $\bar{\bar{\mathcal{R}}}_i$ such that $\mathbf{z}_\nu - \mathbf{z}_{\nu-1} \in S$ for all $\nu \geq 2$, then

$$f(\mathbf{z}_j) = f(\mathbf{z}_1)g(\mathbf{z}_1, \mathbf{z}_j)$$

for all $j \geq 1$. The statement is clearly true if $j = 1$. Suppose inductively that $f(\mathbf{z}_{j-1}) = f(\mathbf{z}_1)g(\mathbf{z}_1, \mathbf{z}_{j-1})$. Since $\mathbf{z}_j - \mathbf{z}_{j-1} \in S$, it follows by (9) that $f(\mathbf{z}_j) = f(\mathbf{z}_{j-1})g(\mathbf{z}_{j-1}, \mathbf{z}_j)$. Hence, $f(\mathbf{z}_j) = f(\mathbf{z}_1)g(\mathbf{z}_1, \mathbf{z}_{j-1})g(\mathbf{z}_{j-1}, \mathbf{z}_j)$. Using the fact that $\prod_i^b \prod_i^c = \prod_i^c$, it is easily seen that $f(\mathbf{z}_j) = f(\mathbf{z}_1)g(\mathbf{z}_1, \mathbf{z}_j)$, as was to be shown. Since \mathcal{R}'_i is S -path connected in $\bar{\bar{\mathcal{R}}}'_i$, it follows that $f(\mathbf{z}_2) = f(\mathbf{z}_1)g(\mathbf{z}_1, \mathbf{z}_2)$ for all \mathbf{z}_1 and $\mathbf{z}_2 \in \mathcal{R}'_i$.

Let $\mathbf{z}_0 \in \mathcal{R}'_i$. We claim that for all $\mathbf{z} \in \mathcal{R}_i$ such that $D(\mathbf{z}) \neq 0$,

$$f(\mathbf{z}) = f(\mathbf{z}_0)g(\mathbf{z}_0, \mathbf{z}).$$

If $\mathbf{z} \in \mathcal{R}'_i$, we're done. If $\mathbf{z} \notin \mathcal{R}'_i$, then, since $D(\mathbf{z}) \neq 0$, we must have $C(\mathbf{z}) = 0$. In this case (9) implies $f(\mathbf{z}) = 0$. But $g(\mathbf{z}_0, \mathbf{z}) = 0$, so $f(\mathbf{z}) = f(\mathbf{z}_0)g(\mathbf{z}_0, \mathbf{z})$.

□

Corollary 3.7.2 *Let f be an honest hypergeometric term on \mathbb{Z}^k . Then there exist relatively prime polynomials C and $D \in K[\mathbf{z}]$ (as in Theorem 3.7.1) and a*

finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero and for each region \mathcal{R}_ℓ there exist a finite set $V \subset \mathbb{Z}^k$ and univariate polynomials a_v and $b_v \in K[z]$ and an integer n_v for each $v \in V$ such that, for all $z \in \mathcal{R}_\ell$,

- (1) $f(z) = \frac{C(z)}{D(z)} \prod_{v \in V} \prod_{j=1}^{v \cdot z + n_v} \frac{a_v(j)}{b_v(j)}$ for all $z \in \mathcal{R}_\ell$ for which $D(z) \neq 0$,
- (2) for all $v \in V$, $v \cdot z + n_v$ is a positive integer, and
- (3) for all $v \in V$ and j , $1 \leq j \leq v \cdot z + n_v$, $a_v(j) \neq 0$ and $b_v(j) \neq 0$.

Corollary 3.7.2 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

Proof. This follows from Theorem 3.7.1 and Lemma 3.5.4. The product

$$\prod_{v \in V} \prod_j^{z \cdot v} \frac{a_v(j)}{b_v(j)}$$

that occurs in the expression for $f(z)$ in Theorem 3.7.1 is a weakly factorial hypergeometric term on the region \mathcal{R}_i . By lemma 3.5.4, the region \mathcal{R}_i can be divided into polyhedral subregions such that the weakly factorial hypergeometric term is factorial on each subregion. \square

Corollary 3.7.3 *Let f be an honest hypergeometric term on \mathbb{Z}^k over a field K that is algebraically closed. Then there exist relatively prime polynomials C and $D \in K[z]$ (as in Theorem 3.7.1) and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\gamma \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that*

- (1) for all $z \in \mathcal{R}_\ell$ such that $D(z) \neq 0$,

$$f(z) = \gamma_1^{z_1} \dots \gamma_k^{z_k} \frac{C(z)}{D(z)} \frac{\prod_{i=1}^p (m_i)_{v_i \cdot z + r_i}}{\prod_{j=1}^q (n_j)_{w_j \cdot z + s_j}};$$

- (2) for all i and j and $z \in \mathcal{R}_\ell$, $v_i \cdot z + r_i$ and $w_j \cdot z + s_j$ are positive;
- (3) the Pochhammer symbols occurring in the products are nonzero.

Corollary 3.7.3 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

Proof. This corollary is an immediate consequence of Corollary 3.7.2 and Lemma 3.5.2. \square

Chapter 4

A Holonomic Hypergeometric Term Is Piecewise Proper

The theory of holonomic systems was developed by Bernstein [2] and applied to the theory of hypergeometric identities by Zeilberger [16]. In [15] Wilf and Zeilberger developed a proof theory for multisums that applies only to *proper* hypergeometric terms. Wilf and Zeilberger conjectured [15] that a hypergeometric term is holonomic if and only if it is proper. We consider the discrete case of their conjecture and interpret it to mean roughly that a hypergeometric term on \mathbb{Z}^k is holonomic if and only if it is *piecewise proper*. We use only elementary facts about holonomic functions that can be found in [16] and [3].

Definition 4.0.1 A hypergeometric term f on \mathbb{Z}^k over an field K is *piecewise proper* if there exist a polynomial $C \in K[\mathbf{z}]$ and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\boldsymbol{\gamma} \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that

(1) for all $\mathbf{z} \in \mathcal{R}_\ell$,

$$f(\mathbf{z}) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} C(\mathbf{z}) \frac{\prod_{i=1}^p (m_i)_{\mathbf{v}_i \cdot \mathbf{z} + r_i}}{\prod_{j=1}^q (n_j)_{\mathbf{w}_j \cdot \mathbf{z} + s_j}};$$

- (2) for all i and j and $\mathbf{z} \in \mathcal{R}_\ell$, $\mathbf{v}_i \cdot \mathbf{z} + r_i$ and $\mathbf{w}_j \cdot \mathbf{z} + s_j$ are positive;
- (3) the Pochhammer symbols occurring in the products are nonzero.

Our interpretation of Wilf and Zeilberger's conjecture is Theorem 4.0.2.

Theorem 4.0.2 A holonomic hypergeometric term f on \mathbb{Z}^k over an algebraically closed field K is *piecewise proper*. Conversely, if f is *piecewise proper* then there

exists a holonomic function g such that $f = g$ a.e.

The following example shows the necessity of introducing sets of measure zero and polyhedral regions to the conjecture. Let

$$f(z_1, z_2) = \begin{cases} (z_1 - z_2 + 1)(z_1 - z_2 - 1) & \text{if } z_1 \neq z_2 \\ g(z_1, z_2) & \text{if } z_1 = z_2, \end{cases}$$

where g is an arbitrary holonomic function. Letting

$$p(z_1, z_2) = (z_1 - z_2 + 1)(z_1 - z_2 - 1),$$

it's easily verified that

$$p^{e_1} f = p f^{e_1} \text{ and } p^{e_2} f = p f^{e_2}$$

regardless of the choice of g , so f is a hypergeometric term. Furthermore, using the fact that the characteristic function of a half-space is holonomic, it's easily seen that f is holonomic:

$$f = \chi(z_1 > z_2)p + \chi(z_1 < z_2)p + \chi(z_1 \geq z_2)\chi(z_1 \leq z_2)g,$$

where $\chi(\text{relation}(z_1, z_2))$ is the characteristic function of $\{(z_1, z_2) : \text{relation}(z_1, z_2)\}$. Each of the relations in the equation defines a half-plane, so each characteristic function is holonomic. Sums and products of holonomic functions are holonomic, so f is holonomic. But f is clearly not proper hypergeometric for arbitrary g .

The function f is, however, *piecewise proper*. The set of measure zero is the line $z_1 = z_2$, and the polyhedral regions are the half-spaces $z_1 > z_2$ and $z_1 < z_2$.

The ‘conversely’ part of Theorem 4.0.2 follows by the arguments of [16]. Thus to prove the conjecture, we need only prove the first part, which is the content of Theorem 4.1.10.

Theorem 4.1.10 A holonomic hypergeometric term f on \mathbb{Z}^k over an algebraically closed field K is piecewise proper: there exist a polynomial $C \in K[\mathbf{z}]$ (as in Theorem 4.1.8) and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the

union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\gamma \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that

(1) for all $\mathbf{z} \in \mathcal{R}_\ell$,

$$f(\mathbf{z}) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} C(\mathbf{z}) \frac{\prod_{i=1}^p (m_i)_{\mathbf{v}_i \cdot \mathbf{z} + r_i}}{\prod_{j=1}^q (n_j)_{\mathbf{w}_j \cdot \mathbf{z} + s_j}};$$

(2) for all i and j and $\mathbf{z} \in \mathcal{R}_\ell$, $\mathbf{v}_i \cdot \mathbf{z} + r_i$ and $\mathbf{w}_j \cdot \mathbf{z} + s_j$ are positive;

(3) the Pochhammer symbols occurring in the products are nonzero.

Theorem 4.1.10 and Theorem 4.0.2 apply to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6. We define a function f on a polyhedral region $\mathcal{R} \subset \mathbb{Z}^k$ to be holonomic if the function g on \mathbb{Z}^k is holonomic, where

$$g(\mathbf{z}) = \begin{cases} f(\mathbf{z}) & \text{if } \mathbf{z} \in \mathcal{R} \\ 0 & \text{otherwise.} \end{cases}$$

4.1 Proof of Theorem 4.1.10

We require the following lemmas.

Lemma 4.1.1 *If f is holonomic and nondegenerate then there exists a finite set $V \subset \mathbb{Z}^k$ such that $f(\mathbf{z}) \neq 0$ implies $f(\mathbf{z} + \mathbf{v}) \neq 0$ for some $\mathbf{v} \in V$.*

Let E_i be the shift operator for z_i . Thus, $\mathbf{E}^{\mathbf{v}} f(\mathbf{z}) = E_1^{v_1} \cdots E_k^{v_k} f(\mathbf{z}) = f(\mathbf{z} + \mathbf{v})$.

Proof. Since f is holonomic, by Lemma 4.1 of [16] or 1.5 of [15], for each $i \in \{1, \dots, k\}$ there exists a nonzero operator $L_i(z_i, \mathbf{E})$ that annihilates f ; $L_i(z_i, \mathbf{E}) = \sum_{\mathbf{v} \in V_i} c_{i,\mathbf{v}}(z_i) \mathbf{E}^{\mathbf{v}}$ for some finite set $V_i \subset \mathbb{Z}^k$ and nonzero univariate polynomials $c_{i,\mathbf{v}} \in K[z]$. The point is that the operator L_i is free of all the variables z_1, \dots, z_k except z_i . Since $L_i f = 0$ implies $\mathbf{E}^{\mathbf{v}} L_i f = 0$, we may assume that $\mathbf{0} \in V_i$.

Let $b_i = c_{i,\mathbf{0}}$. Thus b_i is nonzero and

$$b_i(z_i) f(\mathbf{z}) = - \sum_{\substack{\mathbf{v} \in V_i \\ \mathbf{v} \neq \mathbf{0}}} c_{i,\mathbf{v}}(z_i) f(\mathbf{z} + \mathbf{v}).$$

Hence $b_i(z_i)f(\mathbf{z}) \neq 0$ implies $f(\mathbf{z} + \mathbf{v}) \neq 0$ for some $\mathbf{v} \in V_i$. Letting $\bar{V} = V_1 \cup \dots \cup V_k$ and $S = \{\mathbf{z}: b_i(z_i) = 0 \text{ for } 1 \leq i \leq k\}$, it follows that if $f(\mathbf{z}) \neq 0$ and $\mathbf{z} \notin S$, then $f(\mathbf{z} + \mathbf{v}) \neq 0$ for some $\mathbf{v} \in \bar{V}$. Of course, the set S is finite since each of the b_i has a finite number of roots. Since f is nondegenerate, there exists $\mathbf{z}_0 \in \mathbb{Z}^k$ such that $f(\mathbf{z}_0) \neq 0$ and $\mathbf{z}_0 \notin S$. Finally, let $V = \bar{V} \cup \{\mathbf{z}_0 - \mathbf{z}: \mathbf{z} \in S\}$. \square

Lemma 4.1.2 *If f is holonomic on \mathbb{Z}^k and f is a zero divisor, then $f = 0$ a.e.*

Proof. We prove by induction on the dimension k and the total degree d of p that if f is holonomic on \mathbb{Z}^k and $pf = 0$ a.e. then $f = 0$ a.e. The statement is clearly true if either $k = 1$ or $d = 1$ since in either case p is simple: by Lemma 3.2.3, $p(\mathbf{z}) \neq 0$ except on a set of measure zero, so $f = 0$ a.e.

We assume that the lemma is true if either the dimension is less than k or the degree is less than d and prove that it is true if the dimension is k and the degree is d . Suppose, to the contrary, there exists a polynomial $p \in K[\mathbf{z}]$ of degree d and a holonomic function f on \mathbb{Z}^k such that $pf = 0$ but f is nondegenerate. By Lemma 2.5.3, there exists a finite set V such that $f(\mathbf{z}) \neq 0$ implies $f(\mathbf{z} + \mathbf{v}) \neq 0$ for some $\mathbf{v} \in V$. Let $\mathbf{x} = (x_1, \dots, x_k)$ where the x_i are indeterminates. Let $g = \sum_{\mathbf{v} \in V} f f^{\mathbf{v}} \mathbf{x}^{\mathbf{v}}$. If $f(\mathbf{z}) \neq 0$, then $g(\mathbf{z}) \neq 0$, so g is nondegenerate. Since V is finite, $f f^{\mathbf{v}} = 0$ a.e. for each $\mathbf{v} \in V$ implies $g = 0$ a.e. Thus for at least one $\mathbf{v} \in V$, $f f^{\mathbf{v}}$ is nondegenerate. If $p \neq p^{\mathbf{v}}$, then $q = p - p^{\mathbf{v}}$ is of lower degree and $q f f^{\mathbf{v}} = f^{\mathbf{v}}(pf) - f(pf)^{\mathbf{v}} = 0$ a.e. But $f f^{\mathbf{v}}$ is holonomic, contradicting the assumption that the statement is true if the degree is less than d and the dimension is k .

If $p = p^{\mathbf{v}}$ we reduce the dimension. If h is holonomic on \mathbb{Z}^k , and M is an invertible $k \times k$ integer matrix, then $g(\mathbf{z}) = h(M\mathbf{z})$ is also holonomic. Thus, by a change of variable we may assume that $p = p^{\mathbf{e}_k}$, and hence p is free of z_k .

Let x and y be indeterminates, and let $K[[x, y]]$ be the ring of formal power

series in x and y . Define the function $F: \mathbb{Z}^{k-1} \rightarrow K[[x, y]]$ by

$$F(z_1, \dots, z_{k-1}) = \sum_{z_k \geq 0} f(z_1, \dots, z_k) x^{z_k} + \sum_{z_k < 0} f(z_1, \dots, z_k) y^{-z_k}.$$

By Proposition 3.4 of [16], F is holonomic in z_1, \dots, z_{k-1} . But $pF = 0$, and therefore $F = 0$ a.e. by the assumption that the statement is true for dimension $k-1$ and degree d . The function F is constructed so that the formal power series $F(z_1, \dots, z_{k-1}) = 0$ only if $f(z_1, \dots, z_k) = 0$ for all $z_k \in \mathbb{Z}$. Using the fact that $F = 0$ a.e. in \mathbb{Z}^{k-1} , it is easily seen then that $f = 0$ a.e. in \mathbb{Z}^k , contrary to assumption. \square

Corollary 4.1.3 *A holonomic hypergeometric term is honest.*

Proof. By Lemma 3.1.3 there exist \bar{A}_v and \bar{B}_v such that $\bar{A}_v f = \bar{B}_v f^v$. Let $\bar{A}_v = pA_v$ and $\bar{B}_v = pB_v$ where p , A_v , and B_v are polynomials and A_v and B_v are relatively prime. Then $p(A_v f - B_v f^v) = 0$. But $g = A_v f - B_v f^v$ is holonomic and a zero divisor, so by Lemma 4.1.2, $g = 0$ a.e. Hence, $A_v f = B_v f^v$ a.e. \square

Lemma 4.1.4 *Let f be a hypergeometric term on \mathbb{Z}^k that is not a zero divisor, and let R_v be the term ratio of f in the direction v . Let $V \subset \mathbb{Z}^k$ be a finite set and let $c_v \in K[z]$ be given for each $v \in V$. If $\sum_{v \in V} c_v f^v = 0$ a.e., then $\sum_{v \in V} c_v R_v = 0$. If f is also holonomic, then the converse is true.*

Proof. Since f is a hypergeometric term, by Lemma 3.1.3 there exist polynomials A_v and B_v for each $v \in \mathbb{Z}^k$ such that $A_v f = B_v f^v$. Since $\sum_{v \in V} c_v f^v = 0$ a.e., $(\prod_{w \in V} B_w) \sum_{v \in V} c_v f^v = 0$ a.e., hence $\sum_{v \in V} c_v (\prod_{w \in V \setminus \{v\}} B_w) B_v f^v = 0$ a.e., hence

$$\left(\sum_{v \in V} c_v \left(\prod_{w \in V \setminus \{v\}} B_w \right) A_v \right) f = 0 \text{ a.e. .}$$

Since f is not a zero divisor,

$$\sum_{v \in V} c_v \left(\prod_{w \in V \setminus \{v\}} B_w \right) A_v = 0.$$

Dividing by $\prod_{\mathbf{w} \in V} B_{\mathbf{w}}$ it follows that $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} A_{\mathbf{v}} / B_{\mathbf{v}} = 0$. By Lemma 3.1.5, $R_{\mathbf{v}} = A_{\mathbf{v}} / B_{\mathbf{v}}$. Hence $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} R_{\mathbf{v}} = 0$.

Conversely, assuming f is holonomic, if $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} R_{\mathbf{v}} = 0$, then multiplying by $(\prod_{\mathbf{w} \in V} B_{\mathbf{w}}) f$, it follows that $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} \left(\prod_{\mathbf{w} \in V \setminus \{\mathbf{v}\}} B_{\mathbf{w}} \right) A_{\mathbf{v}} f = 0$. Hence,

$$\sum_{\mathbf{v} \in V} c_{\mathbf{v}} \left(\prod_{\mathbf{w} \in V \setminus \{\mathbf{v}\}} B_{\mathbf{w}} \right) B_{\mathbf{v}} f^{\mathbf{v}} = 0,$$

and hence,

$$\left(\prod_{\mathbf{w} \in V} B_{\mathbf{w}} \right) \sum_{\mathbf{v} \in V} c_{\mathbf{v}} f^{\mathbf{v}} = 0.$$

Since f is holonomic, $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} f^{\mathbf{v}}$ is holonomic. But $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} f^{\mathbf{v}}$ is a zero divisor, so by Lemma 4.1.2, $\sum_{\mathbf{v} \in V} c_{\mathbf{v}} f^{\mathbf{v}} = 0$ a.e. \square

Lemma 4.1.5 *Let $V \subset \mathbb{Z}^2$ be finite and let $A_{\mathbf{v}}$ and $B_{\mathbf{v}} \in K[z_1, z_2]$ be products of nonzero simple polynomials for each $\mathbf{v} \in V$. Let $R \in K(z_1, z_2)$ be a rational function such that*

$$\sum_{\mathbf{v} \in V} \frac{A_{\mathbf{v}}}{B_{\mathbf{v}}} R^{\mathbf{v}} = 0.$$

Then the denominator of R is a product of simple polynomials.

Proof. Let $R = P/Q$, where P and Q are relatively prime. Let d be an irreducible divisor of Q . We will show that d is simple. Suppose the contrary. Then the dimension of $\text{rgal } d$ is less than 1 by the corollary of Lemma 2.5.3, that is, the dimension is 0. Thus $d \neq d^{\mathbf{w}}$ for any nonzero $\mathbf{w} \in \mathbb{Z}^2$. By a corollary of Lemma 2.0.2, $d \neq cd^{\mathbf{w}}$ for any nonzero $\mathbf{w} \in \mathbb{Z}^2$ and $c \in K$. Thus $d^{\mathbf{w}}$ divides Q for only finitely many $\mathbf{w} \in \mathbb{Z}^2$. Let \mathbf{w}_0 be the leftmost of the lowest of the \mathbf{w} such that $d^{\mathbf{w}} \mid Q$. In other words, $\mathbf{w}_0 = (x_0, y_0)$ where

$$y_0 = \min\{y \in \mathbb{Z} : d^{(x, y)} \mid Q \text{ for some } x \in \mathbb{Z}\}$$

and

$$x_0 = \min\{x \in \mathbb{Z} : d^{(x, y_0)} \mid Q\}.$$

Similarly, let \mathbf{v}_0 be the leftmost of the lowest vectors in V .

Multiplying the equation

$$\sum_{\mathbf{v} \in V} \frac{A_{\mathbf{v}}}{B_{\mathbf{v}}} R^{\mathbf{v}} = 0$$

by $\prod_{\mathbf{u} \in V} B_{\mathbf{u}} Q^{\mathbf{u}}$, we have

$$(*) \quad \sum_{\mathbf{v} \in V} A_{\mathbf{v}} P^{\mathbf{v}} \prod_{\mathbf{u} \in V \setminus \{\mathbf{v}\}} B_{\mathbf{u}} Q^{\mathbf{u}} = 0.$$

Since $d^{\mathbf{w}_0} \mid Q$, it follows that $d^{\mathbf{v}_0 + \mathbf{w}_0} \mid Q^{\mathbf{v}_0}$. Since $Q^{\mathbf{v}_0}$ appears in every term of the sum except the term for $\mathbf{v} = \mathbf{v}_0$, it follows that $d^{\mathbf{v}_0 + \mathbf{w}_0} \mid A_{\mathbf{v}_0} P^{\mathbf{v}_0} \prod_{\mathbf{u} \in V \setminus \{\mathbf{v}_0\}} B_{\mathbf{u}} Q^{\mathbf{u}}$. Furthermore, since $A_{\mathbf{v}}$ and $B_{\mathbf{v}}$ are simple and d is not and $Q^{\mathbf{v}_0}$ and $P^{\mathbf{v}_0}$ are relatively prime, it follows that $d^{\mathbf{v}_0 + \mathbf{w}_0} \mid Q^{\mathbf{u}}$ for some $\mathbf{u} \in V \setminus \{\mathbf{v}_0\}$. Thus $d^{\mathbf{w}_0 + \mathbf{v}_0 - \mathbf{u}} \mid Q$ for some $\mathbf{u} \in V \setminus \{\mathbf{v}_0\}$, contradicting the definition of \mathbf{w}_0 . \square

Lemma 4.1.6 *Let f be a nondegenerate holonomic hypergeometric term on \mathbb{Z}^k and let $R_{\mathbf{v}}$ be the term ratio of f in the direction \mathbf{v} for each $\mathbf{v} \in \mathbb{Z}^k$. Let*

$$R_{\mathbf{w}}(z) = \frac{C(z + \mathbf{w})}{C(z)} \frac{D(z)}{D(z + \mathbf{w})} \prod_{\mathbf{v} \in V} \prod_j^{v \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(z \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(z \cdot \mathbf{v} + j)}$$

as in Theorem 3.1.8. Then D is a product of simple polynomials.

Proof. By Lemma 4.1 of [16], for each $i, j \in \{1, \dots, k\}$ there exists a nonzero operator

$$L(E_i, E_j, z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_k)$$

such that L is free of all E s except E_i and E_j , and L is free of z_j . Thus

$$\sum_{\mathbf{w} \in W} \alpha_{\mathbf{w}} f^{\mathbf{w}} = 0,$$

where W is a finite subset of the subspace of \mathbb{Z}^k generated by \mathbf{e}_i and \mathbf{e}_j . Furthermore, since each polynomial $\alpha_{\mathbf{w}}$ is free of z_j , each $\alpha_{\mathbf{w}}$ is simple over the field $K_{i,j} = K(\{z_1, \dots, z_k\} \setminus \{z_i, z_j\})$.

By Lemma 4.1.4, $\sum_{\mathbf{w} \in W} \alpha_{\mathbf{w}} R_{\mathbf{w}} = 0$. Hence

$$\sum_{\mathbf{w} \in W} \alpha_{\mathbf{w}} \frac{C^{\mathbf{w}}}{C} \frac{D}{D^{\mathbf{w}}} \frac{\bar{A}_{\mathbf{w}}}{\bar{B}_{\mathbf{w}}} = 0,$$

where

$$\frac{\bar{A}_{\mathbf{w}}}{\bar{B}_{\mathbf{w}}} = \prod_{\mathbf{v} \in V} \prod_j^{v \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}.$$

Hence

$$\sum_{\mathbf{w} \in W} \alpha_{\mathbf{w}} \frac{\bar{A}_{\mathbf{w}}}{\bar{B}_{\mathbf{w}}} \frac{C^{\mathbf{w}}}{D^{\mathbf{w}}} = 0,$$

and hence

$$\sum_{\mathbf{w} \in W} \frac{A_{\mathbf{w}}}{B_{\mathbf{w}}} \frac{C^{\mathbf{w}}}{D^{\mathbf{w}}} = 0,$$

where $A_{\mathbf{w}}$ and $B_{\mathbf{w}}$ are the products of polynomials that are simple over $K_{i,j}$.

It follows by Lemma 4.1.5 that any irreducible divisor d of D is simple over $K_{i,j}$. Hence, by Corollary 2.5.4, the dimension of $\text{rgal}(d, (z_i, z_j), K_{i,j})$ is at least 1. Let $\mathbb{Q}_{i,j}$ be the subspace of \mathbb{Q}^k generated by \mathbf{e}_i and \mathbf{e}_j . Since $\text{rgal}(d, (z_i, z_j), K_{i,j})$ is isomorphic to $\mathbb{Q}_{i,j} \cap \text{rgal } d$, the dimension of $\mathbb{Q}_{i,j} \cap \text{rgal } d$ is at least 1.

We show that d is simple in z_1, \dots, z_k , that is, the dimension of $\text{rgal } d$ is at least $k - 1$. Let \mathbb{Q}_i be the subspace of \mathbb{Q}^k generated by \mathbf{e}_i . If $\mathbb{Q}_i \subset \text{rgal } d$ for each $i \in \{1, \dots, k\}$, then $\text{rgal } d = \mathbb{Q}^k$. Hence the dimension is k . Otherwise, say for concreteness that $\mathbb{Q}_1 \not\subset \text{rgal } d$. Then for each $j \in \{2, \dots, k\}$ there exists $\mathbf{u}_j \in \mathbb{Q}_{1,j} \cap \text{rgal } d$ such that $\mathbf{u}_j \notin \mathbb{Q}_1$. It's easily seen that $\mathbf{u}_2, \dots, \mathbf{u}_k$ are independent, so the dimension of $\text{rgal } d$ is at least $k - 1$. \square

Lemma 4.1.7 *Let f be a holonomic hypergeometric term on \mathbb{Z}^k and let $R_{\mathbf{v}}$ be the term ratio of f in the direction \mathbf{v} for each $\mathbf{v} \in \mathbb{Z}^k$. There exist a polynomial C , a finite set $V \subset \mathbb{Z}^k$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}}$ for each $\mathbf{v} \in V$ such that*

$$R_{\mathbf{w}}(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{w})}{C(\mathbf{z})} \prod_{\mathbf{v} \in V} \prod_j^{v \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}.$$

Proof. By Theorem 3.1.8 we can express $R_{\mathbf{w}}(\mathbf{z})$ as

$$R_{\mathbf{w}}(\mathbf{z}) = \frac{C(\mathbf{z} + \mathbf{w})}{C(\mathbf{z})} \frac{D(\mathbf{z})}{D(\mathbf{z} + \mathbf{w})} \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{v} \cdot \mathbf{w}} \frac{a_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)}{b_{\mathbf{v}}(\mathbf{z} \cdot \mathbf{v} + j)},$$

and by Lemma 4.1.6 the irreducible divisors of D are simple. The result follows by noting that

$$\frac{d(\mathbf{v} \cdot \mathbf{z})}{d(\mathbf{v} \cdot (\mathbf{z} + \mathbf{w}))} = \prod_j^{\mathbf{v} \cdot \mathbf{w}} \frac{d(\mathbf{v} \cdot \mathbf{z} + j)}{d(\mathbf{v} \cdot \mathbf{z} + 1 + j)},$$

and hence the divisors of D can be absorbed into the ‘factorial part’. \square

Theorem 4.1.8 *Let f be a holonomic hypergeometric term on \mathbb{Z}^k . Then there exist a polynomial C , a finite set $V \subset \mathbb{Z}^k$, and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ for each $\mathbf{v} \in V$ (as in Lemma 4.1.7) and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_m$ such that*

- (1) \mathbb{Z}^k is the union of the \mathcal{R}_i and a set of measure zero;
- (2) for each $i \in \{1, \dots, m\}$ there exists $\mathbf{z}_0 \in \mathcal{R}_i$ such that $C(\mathbf{z}_0) \neq 0$, and for all $\mathbf{z} \in \mathcal{R}_i$,

$$f(\mathbf{z}) = f(\mathbf{z}_0) \frac{C(\mathbf{z})}{C(\mathbf{z}_0)} \prod_{\mathbf{v} \in V} \prod_j^{\mathbf{z} \cdot \mathbf{v}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)}.$$

- (3) all the terms $a_{\mathbf{v}}(j)$ and $b_{\mathbf{v}}(j)$ occurring in the product are nonzero.

Theorem 4.1.8 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

Proof. The theorem is an immediate consequence of Theorem 3.7.1 and Lemma 4.1.7. \square

Corollary 4.1.9 *Let f be an holonomic hypergeometric term on \mathbb{Z}^k . Then there exist a polynomials $C \in K[z]$ and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero and for each region \mathcal{R}_ℓ there exist a finite set $V \subset \mathbb{Z}^k$ and univariate polynomials $a_{\mathbf{v}}$ and $b_{\mathbf{v}} \in K[z]$ and an integer $n_{\mathbf{v}}$ for each $\mathbf{v} \in V$ such that, for all $\mathbf{z} \in \mathcal{R}_\ell$,*

$$(1) f(\mathbf{z}) = C(\mathbf{z}) \prod_{\mathbf{v} \in V} \prod_{j=1}^{\mathbf{v} \cdot \mathbf{z} + n_{\mathbf{v}}} \frac{a_{\mathbf{v}}(j)}{b_{\mathbf{v}}(j)},$$

- (2) for all $\mathbf{v} \in V$, $\mathbf{v} \cdot \mathbf{z} + n_{\mathbf{v}}$ is a positive integer, and
 (3) for all $\mathbf{v} \in V$ and j , $1 \leq j \leq \mathbf{v} \cdot \mathbf{z} + n_{\mathbf{v}}$, $a_{\mathbf{v}}(j) \neq 0$ and $b_{\mathbf{v}}(j) \neq 0$.

Corollary 4.1.9 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

Proof. This corollary is an immediate consequence of Corollary 3.7.2 and Lemma 4.1.7. \square

Theorem 4.1.10 *A holonomic hypergeometric term f on \mathbb{Z}^k over an algebraically closed field K is piecewise proper: there exist a polynomial $C \in K[\mathbf{z}]$ (as in Theorem 4.1.8) and a finite number of polyhedral regions $\mathcal{R}_1, \dots, \mathcal{R}_L$ such that \mathbb{Z}^k is the union of the \mathcal{R}_ℓ and a set of measure zero, and for each region \mathcal{R}_ℓ there exist a vector $\boldsymbol{\gamma} \in K^k$, constants $m_1, \dots, m_p, n_1, \dots, n_q \in K$, vectors $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q \in \mathbb{Z}^k$, and integers $r_1, \dots, r_p, s_1, \dots, s_q$ such that*

(1) for all $\mathbf{z} \in \mathcal{R}_\ell$,

$$f(\mathbf{z}) = \gamma_1^{z_1} \cdots \gamma_k^{z_k} C(\mathbf{z}) \frac{\prod_{i=1}^p (m_i)_{\mathbf{v}_i \cdot \mathbf{z} + r_i}}{\prod_{j=1}^q (n_j)_{\mathbf{w}_j \cdot \mathbf{z} + s_j}};$$

- (2) for all i and j and $\mathbf{z} \in \mathcal{R}_\ell$, $\mathbf{v}_i \cdot \mathbf{z} + r_i$ and $\mathbf{w}_j \cdot \mathbf{z} + s_j$ are positive;
 (3) the Pochhammer symbols occurring in the products are nonzero.

Theorem 4.1.10 applies to hypergeometric terms on polyhedral regions other than \mathbb{Z}^k via Lemma 3.4.6.

Proof. This theorem is an immediate consequence of Corollary 4.1.9 and Lemma 3.5.2. \square

Chapter 5

A Solution to a Problem of Cameron On Sum-free Complete Sets

5.1 Introduction

For any subsets A and B of an additive group G , define $A + B = \{a + b : a \in A \text{ and } b \in B\}$ and $-A = \{-a : a \in A\}$. A subset S of G is said to be sum-free, complete, and symmetric respectively if $S + S \subset S^c$, $S + S \supset S^c$, and $S = -S$. Hence, S is sum-free and complete if and only if $S + S = S^c$.

Cameron observed that for any sufficiently small modulus m , every sum-free complete set in $\mathbb{Z}/m\mathbb{Z}$ is also symmetric. In fact, Calkin found that $m = 36$ is the smallest modulus for which there is a sum-free complete set that is not symmetric [7]. Cameron asked if there exists such a set for all sufficiently large moduli [7]. We answer Cameron's question by showing there exists such a set for all moduli greater than or equal to 890626.

We also show that every sum-free complete set in $\mathbb{Z}/m\mathbb{Z}$ that is not symmetric can be used to construct a counterexample to a conjecture of J.H. Conway. Conway conjectured that for any finite set S of integers, $|S + S| \leq |S - S|$. Conway's conjecture was disproved by Marica [9]. Later Stein showed how to make the ratio $|S + S|/|S - S|$ arbitrarily large [12]. We show that if S is sum-free and complete modulo m but not symmetric, then $|S + S| > |S - S|$; hence, S is a counterexample to a modular version of Conway's conjecture. Further, we show that if $S' \subset \mathbb{Z}$ is a certain set derived from S , then $|S' + S'| > |S' - S'|$; hence, S' is a counterexample to Conway's conjecture proper.

The history of sum-free sets begins with Schur who showed that the positive integers can not be partitioned into finitely many sum-free sets [11]. Sum-free sets have been used to find lower bounds for Ramsey numbers (see pp. 28, 128, 264 in [13]). Cameron describes some applications of sum-free sets and poses several problems [5],[6],[7]. George Andrews observed that sum-free complete sets play a role in partition identities (personal communication). For example, the set $\{1, 4\} \in \mathbb{Z}/5\mathbb{Z}$, which arises in the Rogers-Ramanujan Identities (p. 109 in [1]), is sum-free, complete, and symmetric. Calkin showed that the number of sum-free sets contained within the first n integers is $o(2^{n(1/2+\epsilon)})$ for every $\epsilon > 0$ [4].

5.2 Cameron's problem

For any $S \subset \mathbb{Z}$ and $a, b \in \mathbb{Z} \cup \{-\infty, \infty\}$ define $S_a^b = \{s \in S : a \leq s \leq b\}$. The following sets S_1, \dots, S_5 are used as building blocks in the construction of infinite families of modular sum-free complete sets that are not symmetric. Define

$$\begin{aligned} S_1 &= (-3 + 5\mathbb{Z})_{-\infty}^{-354} \cup (F_1)_{-353}^{353} \cup (3 + 5\mathbb{Z})_{354}^{\infty}, \\ S_2 &= (-1 + 5\mathbb{Z})_{-\infty}^{-192} \cup (F_2)_{-191}^{191} \cup (1 + 5\mathbb{Z})_{192}^{\infty}, \\ S_3 &= (-4 + 5\mathbb{Z})_{-\infty}^{-185} \cup (F_3)_{-184}^{184} \cup (4 + 5\mathbb{Z})_{185}^{\infty}, \\ S_4 &= (-2 + 5\mathbb{Z})_{-\infty}^{-253} \cup (F_4)_{-252}^{252} \cup (2 + 5\mathbb{Z})_{253}^{\infty}, \quad \text{and} \\ S_5 &= (-1 + 3\mathbb{Z})_{-\infty}^{-95} \cup (F_5)_{-94}^{94} \cup (1 + 3\mathbb{Z})_{95}^{\infty}, \end{aligned}$$

where

$$\begin{aligned} F_1 &= \{-6, 3\} \cup \pm\{1, 8, 13, 17, 22, 27, 38, 42, 53, 58, 62, 67, 72, 74, 86, 88, 93, 98, 107, \\ &\quad 117, 119, 121, 133, 137, 142, 147, 152, 168, 173, 178, 182, 187, 192, 197, 208, \\ &\quad 213, 218, 222, 227, 232, 243, 248, 253, 288, 293, 298, 323, 328, 333, 338\}, \\ F_2 &= \{-6, 3\} \cup \pm\{1, 8, 13, 17, 22, 27, 38, 42, 53, 58, 62, 72, 74, 86, 88, 93, 109, 119, \\ &\quad 121, 156, 166\}, \end{aligned}$$

$$F_3 = \{-6, 3\} \cup \pm\{1, 8, 13, 17, 22, 27, 38, 42, 53, 58, 62, 67, 72, 74, 86, 88, 93, 98, 119, \\ 149, 154, 159, 164, 169\},$$

$$F_4 = \{-6, 3\} \cup \pm\{1, 8, 13, 17, 22, 27, 38, 42, 53, 58, 62, 67, 72, 74, 86, 88, 93, 98, 107, \\ 117, 119, 121, 133, 137, 142, 147, 152, 182, 187, 192, 197, 202, 222, 227, 232, \\ 237, 247\}, \quad \text{and}$$

$$F_5 = \{-1, 2\} \cup \pm\{5, 8, 14, 17, 29, 40, 44, 47, 67, 70, 79, 82, 85\}.$$

Lemma 5.2.1 reduces to a computation the problem of determining if a set of the form of S_1, \dots, S_5 is sum-free and complete.

Lemma 5.2.1 *Let $S = (A + m\mathbb{Z})_{-\infty}^{-a} \cup F_{-a+1}^{b-1} \cup (B + m\mathbb{Z})_b^{\infty}$ where a, b , and m are positive integers and A, F , and $B \subset \mathbb{Z}$. The set S is sum-free and complete in \mathbb{Z} if and only if*

$$(S^c)_{-2a-2m}^{2b+2m} = (S_{-(2a+b+3m)}^{2b+a+3m} + S_{-(2a+b+3m)}^{2b+a+3m})_{-2a-2m}^{2b+2m}.$$

Proof. Let $i = -(2a + 2m)$, $j = 2b + 2m$, $k = -(2a + 2b + 3m)$, and $l = 2b + a + 3m$. We need to show that S is sum-free and complete if and only if $(S^c)_i^j = (S_k^l + S_k^l)_i^j$. We first show that S is sum-free and complete if and only if $(S^c)_i^j = (S + S)_i^j$, and then show that $(S + S)_i^j = (S_k^l + S_k^l)_i^j$. To prove the first assertion it suffices to show that if $(S^c)_i^j = (S + S)_i^j$, then S is sum-free and complete; the implication in the other direction is trivial.

We show that if $(S^c)_i^j = (S + S)_i^j$, then S is sum-free. Suppose the contrary and let s_1, s_2 , and $s_3 \in S$ be such that $s_1 + s_2 = s_3$ and $|s_3|$ is minimal. By assumption $(S + S)_i^j \subset S^c$, so $s_3 > j$ or $s_3 < i$. We consider the case $s_3 > j$. Let $s'_3 = s_3 - m$. Since $s_3 > j > b + m$, it follows that $s_3 \in (B + m\mathbb{Z})_b^{\infty}$ and $s'_3 \in (B + m\mathbb{Z})_b^{\infty}$. Since $s_3 \geq 2b + 2m$, it follows that $s_1 \geq b + m$ or $s_2 \geq b + m$. We may assume without loss of generality that $s_1 \geq b + m$. Let $s'_1 = s_1 - m$. Since $s_1 \geq b + m$, it follows that $s_1 \in (B + m\mathbb{Z})_b^{\infty}$ and $s'_1 \in (B + m\mathbb{Z})_b^{\infty}$. But $s'_1 + s_2 = s'_3$ and $|s'_3| < |s_3|$, contradicting the minimality of $|s_3|$. The case $s_3 < i$ is similar.

We show that if $(S^c)_i^j = (S+S)_i^j$, then S is complete. Suppose the contrary and let $c \in S^c$ be such that $c \notin S+S$ and $|c|$ is minimal. By assumption, $(S^c)_i^j \subset S+S$, so $c > j$ or $c < i$. We consider the case $c > j$. Let $T = (B+m\mathbb{Z})^c$. Since $c > j > b+m$, it follows that $c \in T_{b+m}^\infty$. Let $c' = c - m$. Since $T = T + m\mathbb{Z}$, it follows that $c' \in T_b^\infty \subset S^c$. By the minimality of $|c|$, it follows that $c' = s_1 + s_2$ where s_1 and $s_2 \in S$. Since $c' > j - m > 2b$, it follows that $s_1 \geq b$ or $s_2 \geq b$. We may assume without loss of generality that $s_1 \geq b$. Let $s'_1 = s_1 + m$. Since $s_1 \geq b$, it follows that $s_1 \in (B+m\mathbb{Z})_b^\infty$ and hence $s'_1 \in (B+m\mathbb{Z})_b^\infty$. But $c = s'_1 + s_2$ contradicting the assumption $c \notin S+S$. The case $c < i$ is similar.

We complete the proof by showing that $(S+S)_i^j = (S_k^l + S_k^l)_i^j$. It suffices to show that $(S+S)_i^j \subset S_k^l + S_k^l$. Let $c \in (S+S)_i^j$ and let s_1 and $s_2 \in S$ be such that $s_1 + s_2 = c$ and $|s_1| + |s_2|$ is minimal. We claim that $s_1, s_2 \in S_k^l$ and, hence, $c \in (S_k^l + S_k^l)_i^j$. Suppose to the contrary $s_1 > l$ or $s_1 < k$. We consider the case $s_1 > l$. Let $s'_1 = s_1 - m$ and $s'_2 = s_2 + m$. Since $s_1 > l > b+m$, it follows that $s_1 \in (B+m\mathbb{Z})_b^\infty$ and $s'_1 \in (B+m\mathbb{Z})_b^\infty$. Since $s_1 > l = 2b+a+3m$ and $c \leq 2b+2m$, it follows that $s_2 = c - s_1 \leq -a - m$. Since $s_2 \leq -a - m$, it follows that $s_2 \in (A+m\mathbb{Z})_{-\infty}^{-a}$ and $s'_2 \in (A+m\mathbb{Z})_{-\infty}^{-a}$. But $s'_1 + s'_2 = c$ and $|s'_1| + |s'_2| < |s_1| + |s_2|$, contradicting the minimality of $|s_1| + |s_2|$. Hence, the claim is true in the case $s_1 > l$. The case $s_1 < k$ is similar. \square

Using Lemma 5.2.1, it is easily shown that the sets S_1, \dots, S_5 are sum free and complete. Taking $F = F_1$, $a = b = 354$, $m = 5$, and $-A = B = \{3\}$ and verifying that $((S_1)^c)_{-718}^{718} = ((S_1)_{-1077}^{1077} + (S_1)_{-1077}^{1077})_{-718}^{718}$, it follows from Lemma 5.2.1 that S_1 is sum-free and complete. The cases corresponding to S_2, \dots, S_5 are proved similarly.

Let $T \subset \mathbb{Z}$ and let n be a positive integer. A sum-free complete set of the form $T + n\mathbb{Z}$ can be identified with a sum-free complete set modulo n ; the subset $T + n\mathbb{Z}$ of \mathbb{Z} is sum-free and complete if and only if the subset $\{t + n\mathbb{Z} : t \in T\}$ of $\mathbb{Z}/n\mathbb{Z}$

is sum-free and complete. Lemma 5.2.2 shows that a sum-free complete set of the form of S_1, \dots, S_5 can be used to construct a sum-free complete set modulo n for all sufficiently large n in an arithmetic progression.

Lemma 5.2.2 *Let $S = (A + m\mathbb{Z})_{-\infty}^{-a} \cup F_{-a+1}^{b-1} \cup (B + m\mathbb{Z})_b^{\infty}$ where a, b and m are positive integers, A, F and $B \subset \mathbb{Z}$, and for some integer d , $d + A + m\mathbb{Z} = B + m\mathbb{Z}$. If S is sum-free and complete in \mathbb{Z} , then $S_{-n/2}^{n/2} + n\mathbb{Z}$ is sum-free and complete in \mathbb{Z} for all n such that $n \geq 2m + 4 \max(a, b) - 2$ and $n \cong d \pmod{m}$.*

Proof. Let n be such that $n \geq 2m + 4 \max(a, b) - 2$ and $n \cong d \pmod{m}$.

We show that $S_{-n/2}^{n/2} + n\mathbb{Z}$ is sum-free. Suppose the contrary and let s_1, s_2 , and $s_3 \in S_{-n/2}^{n/2}$ be such that $s_1 + s_2 \cong s_3 \pmod{n}$. Since $|s_1 + s_2 - s_3| \leq |s_1| + |s_2| + |s_3| \leq 3n/2$ and $s_1 + s_2 - s_3 \neq 0$, it follows that $s_1 + s_2 - s_3 = \pm n$. We consider the case $s_1 + s_2 - s_3 = n$. Let $\text{sg}(1) = \text{sg}(2) = -\text{sg}(3) = 1$. If for all $i \in \{1, 2, 3\}$ either $s_i \in F_{-a+1}^{b-1}$ or $s_i \text{sg}(i) \leq 0$, then $s_1 + s_2 - s_3 \leq 3b < n$, so we may assume there exists $j = 1, 2$, or 3 such that $s_j \notin F_{-a+1}^{b-1}$ and $s_j \text{sg}(j) > 0$. In fact, we may assume without loss of generality that either $j = 1$ or $j = 3$.

We consider the case $j = 1$. Let $s'_1 = s_1 - n$. Since $s_1 \notin F_{-a+1}^{b-1}$ and $s_1 > 0$, it follows that $s_1 \in B + m\mathbb{Z}$ and $s'_1 \in B + m\mathbb{Z} - n$. Since $n \cong d \pmod{m}$, it follows that $m\mathbb{Z} - n = m\mathbb{Z} - d$. Hence, $s'_1 \in B + m\mathbb{Z} - d = A + m\mathbb{Z}$. Since $s_1 \leq n/2$, it follows that $s'_1 \leq -n/2 \leq -a$. Hence, $s'_1 \in (A + m\mathbb{Z})_{-\infty}^{-a} \subset S$. But $s'_1 + s_2 = s_3$, contradicting the assumption that S is sum-free.

The case $j = 3$ is similar. This completes the proof in the case $s_1 + s_2 - s_3 = n$. The case $s_1 + s_2 - s_3 = -n$ is similar. We have shown that $S_{-n/2}^{n/2} + n\mathbb{Z}$ is sum-free.

We show $S_{-n/2}^{n/2} + n\mathbb{Z}$ is complete. Let $c \in S^c$ be such that $|c| \leq n/2$. We need to show that $c \cong t_1 + t_2 \pmod{n}$ for some t_1 and $t_2 \in S_{-n/2}^{n/2}$. Since S is complete, $c = s_1 + s_2$ for some s_1 and $s_2 \in S$. Let s_1 and s_2 be such that $|s_1 - s_2|$ is minimal. If s_1 and s_2 have the same sign, $|s_1| + |s_2| = |c| \leq n/2$. Hence, s_1 and $s_2 \in S_{-n/2}^{n/2}$ and we are done, so assume $s_1 > 0$ and $s_2 < 0$.

We claim that $s_1 < b + m$ or $s_2 > -a - m$. Suppose the contrary and let $s'_1 = s_1 - m$ and $s'_2 = s_2 + m$. Since $s_1 \geq b + m$, it follows that s_1 and $s'_1 \in (B + m\mathbb{Z})_b^\infty$. Hence, $s'_1 \in S$. Similarly, $s'_2 \in S$. But $s'_1 + s'_2 = c$ and $|s'_1 - s'_2| < |s_1 - s_2|$, contradicting the minimality of $|s_1 - s_2|$. Hence, $s_1 < b + m$ or $s_2 > -a - m$ as claimed.

We consider the case $s_2 > -a - m$. If $s_1 \leq n/2$ we are done, so assume $s_1 > n/2$ and let $s'_1 = s_1 - n$. Since $s_1 > n/2 > b$, it follows that $s_1 \in B + m\mathbb{Z}$. Arguing as in the case $j = 1$ above, $s'_1 \in A + m\mathbb{Z}$. Since $c \leq n/2$ and $s_2 > -a - m$, it follows that $s'_1 = c - s_2 - n < -n/2 + a + m$. Since $n \geq 2m + 4a - 2$, it follows that $-n/2 + m + a \leq -a + 1$. Hence, $s'_1 < -a + 1$. Since $s'_1 \leq -a$ and $s'_1 \in A + m\mathbb{Z}$, it follows that $s'_1 \in S$. Since $s_1 > n/2$, it follows that $s'_1 > -n/2$, and hence $s'_1 \in S_{-n/2}^{n/2}$. But $s'_1 + s_2 \cong c \pmod{n}$. The case $s_1 < b + m$ is similar. \square

Theorem 5.2.3 *For all $n \geq 890626$, there exists a sum-free complete set in $\mathbb{Z}/n\mathbb{Z}$ that is not symmetric.*

Proof. Applying Lemma 5.2.2 with $S = S_1$, $A = \{-3\}$, $B = \{3\}$, $F = F_1$, $a = b = 354$, $m = 5$, and $d = 1$, it follows that for every n such that $n \geq 1424$ and $n \cong 1 \pmod{5}$, the set $(S_1)_{-n/2}^{n/2} + n\mathbb{Z}$ is sum-free and complete. Equivalently, for every n such that $n \geq 1426$ and $n \cong 1 \pmod{5}$, the set $T_n = \{s + n\mathbb{Z} : s \in (S_1)_{-n/2}^{n/2}\} \subset \mathbb{Z}/n\mathbb{Z}$ is sum-free and complete. Further, it is clear from the form of S_1 that $3 \in T_n$ and $-3 \notin T_n$. Hence, T_n is not symmetric. Thus, we have shown that there is a sum-free complete set that is not symmetric for all moduli in the set $R_1 = \{m \geq 1426 : m \cong 1 \pmod{5}\}$. Similar arguments using the sets S_2, \dots, S_5 show that there is a sum-free complete set that is not symmetric for all moduli in sets $R_2 = \{m \geq 777 : m \cong 2 \pmod{5}\}$, $R_3 = \{m \geq 748 : m \cong 3 \pmod{5}\}$, $R_4 = \{m \geq 1024 : m \cong 4 \pmod{5}\}$, and $R_5 = \{m \geq 386 : m \cong 2 \pmod{3}\}$.

It is easily seen that if there is a sum-free complete set that is not symmetric for the modulus m , then there is such a set for any modulus that is a multiple of

m . Hence, there is a sum-free complete set that is not symmetric for any modulus with a divisor in the set $R = R_1 \cup R_2 \cdots \cup R_5$.

It remains only to show that for all $n \geq 890626$, n has a divisor in the set R . Let n be greater than or equal to 890626 and write n in the form $n = 5^a b$ where b is not divisible by 5. Since $890626 = 5^4 \cdot 1426 + 1$, either $a \geq 5$ or $b \geq 1426$. If $a \geq 5$, then $5^5 = 3125$ divides n . Since $3125 \in R_5$, it follows that $3125 \in R$. If $b \geq 1426$, then $b \in R$ since the set $R_1 \cup R_2 \cdots \cup R_4$ contains every number greater than or equal to 1426 that is not a multiple of 5. In either case n has a divisor in R . We have shown that for every n greater than or equal to 890626 there is a sum-free complete set in $\mathbb{Z}/n\mathbb{Z}$ that is not symmetric. \square

The condition that $n \geq 890626$ in Theorem 5.2.3 is not sharp and can undoubtedly be considerably reduced.

5.3 Conway's conjecture

The following theorem shows that if S is sum-free and complete but not symmetric modulo m , then S must be a counterexample to a modular version of Conway's conjecture.

Theorem 5.3.1 *If a set S is sum-free and complete but not symmetric modulo m , then $|S + S| > |S - S|$.*

Before beginning the proof of Theorem 5.3.1, it is useful to observe that $S + S \subset S^c$ if and only if $S - S \subset S^c$; there are no solutions $s_1, s_2, s_3 \in S$ to $s_1 + s_2 = s_3$ if and only if there are no solutions to $s_3 - s_2 = s_1$.

Proof. Suppose to the contrary S is sum-free and complete but not symmetric modulo m and $|S + S| \leq |S - S|$. Since S is sum-free, $S + S \subset S^c$ and hence $S - S \subset S^c$. Since $S^c = S + S$, it follows that $S - S \subset S + S$. But $|S + S| \leq |S - S|$, so $S - S = S + S$ and hence $S - S = S^c$. The set $S - S$ is symmetric, hence S^c is symmetric, hence S is symmetric, contrary to assumption. \square

Theorem 5.3.2 shows that if S is sum-free and complete but not symmetric

modulo m , then S can be used to construct a counterexample to Conway's conjecture.

Theorem 5.3.2 *Let $A \subset \mathbb{Z}/m\mathbb{Z}$ be sum-free and complete but not symmetric, let $r = |A - A|/|A + A|$, and let S be the set of integers congruent modulo m to a member of A . For all $n \geq 2m/(1 - r)$, we have $|S_0^n + S_0^n| > |S_0^n - S_0^n|$.*

We require the following lemma for the proof of Theorem 5.3.2.

Lemma 5.3.3 *Let $A, B \subset \mathbb{Z}$, $S = A + m\mathbb{Z}$, and $T = B + m\mathbb{Z}$. For any integers i, j, k , and l such that $j - i \geq m - 1$ and $k - l \geq m - 1$,*

$$(S_i^j + T_k^l)_{i+k+m-1}^{j+l-m+1} = (S + T)_{i+k+m-1}^{j+l-m+1}.$$

Proof. To prove Lemma 5.3.3, we need only show

$$(1) \quad (S_i^j + T_k^l)_{i+k+m-1}^{j+l-m+1} \supset (S + T)_{i+k+m-1}^{j+l-m+1},$$

since the containment in the other direction is trivial. We first consider the case $j - i = k - l = m - 1$. Let $s = j + l - m + 1 = i + k + m - 1$. Since $j - i \geq m - 1$, $S = S_i^j + m\mathbb{Z}$. Similarly, $T = T_k^l + m\mathbb{Z}$. Hence,

$$\begin{aligned} S + T &= (S_i^j + m\mathbb{Z}) + (T_k^l + m\mathbb{Z}) = S_i^j + T_k^l + m\mathbb{Z} \\ &= S_i^j + T_k^l + (m\mathbb{Z}_{-\infty}^{-1} \cup \{0\} \cup m\mathbb{Z}_1^{\infty}) \\ &= (S_i^j + T_k^l + m\mathbb{Z}_{-\infty}^{-1}) \cup (S_i^j + T_k^l) \cup (S_i^j + T_k^l + m\mathbb{Z}_1^{\infty}) \\ &\subset \mathbb{Z}_{-\infty}^{j+l-m} \cup (S_i^j + T_k^l) \cup \mathbb{Z}_{i+k+m}^{\infty} \\ &\subset \mathbb{Z}_{-\infty}^{s-1} \cup (S_i^j + T_k^l) \cup \mathbb{Z}_{s+1}^{\infty}. \end{aligned}$$

It follows that $(S_i^j + T_k^l)_s^s \supset (S + T)_s^s$ which is that statement of (1) for the case $j - i = k - l = m - 1$.

We derive the general case from the previous one. Let $j - i \geq m - 1$, $k - l \geq m - 1$

and define $I = \{(a, b, c, d) : \mathbb{Z}_a^b \subset \mathbb{Z}_i^j, \mathbb{Z}_c^d \subset \mathbb{Z}_k^l, \text{ and } b - a = d - c = m - 1\}$.

$$\begin{aligned}
(S_i^j + T_k^l)_{i+k+m-1}^{j+l-m+1} &\supset \bigcup_{(a,b,c,d) \in I} (S_a^b + T_c^d)_{a+c+m-1}^{b+d-m+1} \\
&\supset \bigcup_{(a,b,c,d) \in I} (S + T)_{a+c+m-1}^{b+d-m+1} \quad \text{by the previous case} \\
&\supset (S + T)_{i+k+m-1}^{j+l-m+1},
\end{aligned}$$

so (1) is proved. \square

Proof. We now prove Theorem 5.3.2. Let $n \geq 2m/(1-r)$. By Theorem 5.3.1, $r < 1$ so $n \geq m - 1$. Hence, by Lemma 5.3.3,

$$(S_0^n + S_0^n)_{m-1}^{2n-m+1} = (S + S)_{m-1}^{2n-m+1} = (S + S)_{m-1}^{m-1+km-1} \supset (S + S)_{m-1}^{m-1+[k]m-1},$$

where $k = (2n - 2m + 3)/m$ and $[k]$ is the greatest integer less than or equal to k . Since $S + S$ is the set of integers congruent modulo m to a member of $A + A$, it follows that $|(S + S)_a^{a+im-1}| = i|A + A|$ for any a and $i \geq 0$. In particular, $|(S + S)_{m-1}^{m-1+[k]m-1}| = [k]|A + A|$. Hence $|S_0^n + S_0^n| > [k]|A + A| > (k-1)|A + A|$.

Since $S_0^n - S_0^n \subset \mathbb{Z}_{-n}^n$, it follows that $S_0^n - S_0^n \subset (S - S)_{-n}^n \subset (S - S)_{-n}^{-n+[j]m-1}$, where $j = (2n+1)/m$ and $[j]$ is the least integer greater than or equal to j . Arguing as before, $|(S - S)_{-n}^{-n+[j]m-1}| = [j]|A - A|$. Hence, $|S_0^n - S_0^n| < [j]|A - A| < (j+1)|A - A|$.

It follows that

$$\frac{|S_0^n + S_0^n|}{|S_0^n - S_0^n|} > \frac{(k-1)|A + A|}{(j+1)|A - A|} = \frac{2n - 3m + 3}{(2n + m + 1)r}.$$

The right side is greater than or equal to 1 if and only if

$$n \geq \frac{(3+r)m + r - 3}{2(1-r)}.$$

Since $r < 1$, a sufficient condition is that $n \geq 2m/(1-r)$. \square

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Corrigenda

- p. 6. “term on \mathbb{Z}^k f ” has been corrected to read “term f on \mathbb{Z}^k ” .
- p. 14. “hence $d^{T_j} \mid Y(S_j)$ ” has been corrected to read “hence $d^{T_i} \mid Y(S_j)$ ” .
- p. 17. “Let A and B be invertible $k \times k$ matrices” has been corrected to read “Let A and B be invertible $r \times r$ and $k \times k$ matrices respectively” .
- p. 25. Near the bottom of the page, “ $r_i =$ ” has been corrected to read “ $R_i =$ ” .
- pp. 4, 36. “ $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} = 0\}$, where $\mathbf{v} \dots \{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} > 0\}$, where \mathbf{v} ” has been corrected to read “ $\{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} = n\}$, where $n \in \mathbb{Z}$ and $\mathbf{v} \dots \{\mathbf{z}: \mathbf{v} \cdot \mathbf{z} > n\}$, where $n \in \mathbb{Z}$ and \mathbf{v} ” .
- p. 38. “degree at most $k - 1$ ” has been corrected to read “degree at most $n - 1$ ”
- p. 41. “ $j - a_j$ ” has been corrected to read “ $j - a_t$ ” .
- p. 42. Both occurrences of “ $z_1 < 1$ ” have been corrected to read “ $z_1 < 0$ ” and “ $j - a_j$ ” has been corrected to read “ $j - a_1$ ” .
- p. 45. “entirely in \mathcal{R}' .” has been corrected to read “entirely in \mathcal{R} .” .
- p. 57. “implies $f(\mathbf{z} + \mathbf{v}) \neq 0$ for some $\mathbf{v} \in V$ ” has been corrected to read “implies $f(\mathbf{z} + \mathbf{v}) \neq 0$ for some $\mathbf{v} \in V_i$ ” .
- p. 58. “ $\sum_{z_k < 0} f(z_1, \dots, z_k) y^{z_k}$ ” has been corrected to read “ $\sum_{z_k < 0} f(z_1, \dots, z_k) y^{-z_k}$ ” .
- p. 61. “there exists $\mathbf{u}_j \in \mathbb{Q}_{i,j} \cap \text{rgal } d$ such that $\mathbf{u}_j \notin \mathbb{Q}_i$.” has been corrected to read “there exists $\mathbf{u}_j \in \mathbb{Q}_{1,j} \cap \text{rgal } d$ such that $\mathbf{u}_j \notin \mathbb{Q}_1$.”

The author was unaware that Theorem 2.8.4 is due to O. Ore [1] in the case of two variables and M. Sato [2] in the general case.

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